Determine genus from Bernardi torsor

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Abstract

This semester I was examining chip firing games with the help of my supervisor, Lilla Tóthmérész to whom I owe a lot of thanks for this report to come true. In this paper we will prove a partial result of one of the open problems in chip firing games, specifically we will show that if we have a cut-free ribbon graph G, whose ribbon structure is unknown, we can determine the topological genus of the graph based on the Bernardi torsor $\{\beta_v(x) : x \in S(G)\}_{v\in V(G)}$, where $S(G)$ is the sandpile group.

This work is inspired from and heavily relies on a recent result of Alex McDonough from [9].

Preliminaries

Björner, Lovász and Shor introduced the concept of chip firing games in 1991 in [4], although different variations were considered earlier as well. Originally it was defined in the following way.

Let $G = (V, E)$ a finite, connected undirected graph and let $c \in \mathbb{N}^V$ a so-called chip configuration. A vertex is *ready to fire* if $c_v \geq \deg(v)$. Whenever a chip fires, it sends across one chip on each edges adjacent to it. Let $L(G)$ denote the Laplace matrix of G. Then the resulting configuration c' from an initial configuration c whenever a vertex v fires can be written as

$$c' = c - L \cdot \chi_v,$$

where χ_v is the characteristic vector of the vertex v. A configuration is said to be *stable*, if $c_v < \deg(v)$, $\forall v \in V$, that is no vertex is ready to fire.

Sandpile group

For further examining one can introduce the *the divisor group* $Div(G)$ of a graph G as follows:

$$Div(G) = \{(x_v)_{v\in V(G)} | x_v \in \mathbb{Z}\}.$$

In other words the group of all possible chip-configurations over the integers. An important subgroup is the *zero-degree divisor group* $Div^0(G)$, defined as

$$Div^0(G) = \left\{(x_v)_{v\in V(G)} | x_v \in \mathbb{Z}, \textstyle\sum_{v\in V(G)} x_v = 0\right\}.$$

This way, we have an initial configuration $x \in \mathbb{Z}^V$ on the vertices, whose overall number of chips is zero. In general let the degree of a configuration be $\sum_{v\in V(G)} x_v$, and $Div^k(G)$ the set of k-degree configuration.

In this setting each vertex can *either fire or borrow (unfire)*, regardless the number of chips they have. This raises a linear equivalence on the configurations as follows:

$$x \sim y \Leftrightarrow \exists z \in \mathbb{Z}^V : y = x - Lz \Leftrightarrow x - y \in im(L),$$

that is if they are reachable from each other via firing and borrowing steps. Let us notice that firing or unfiring vertices does not change the degree of the resulting configurations. Hence, equivalent configurations have the same degree. Factoring with this equivalence $Div^0(G)$ gives the sandpile group $S(G)$:

$$S(G) = \frac{Div^0(G)}{im(L)}.$$

Generally let $S^k(G) = \dfrac{Div^k(G)}{im(L)}$.

Since unfiring a vertex yields the same configuration as firing every other vertex, the following useful lemma holds.

Lemma 1. *Two configuration $x, y \in Div^k(G)$ represents the same element in $S^k(G)$ for any $k \in \mathbb{N}$, if and only if y is reachable from x via firing steps. Consequently $x, y \in Div^0(G)$ represent the same element of $S(G)$ if and only if the all zero configuration is reachable from the configuration $x - y$.*

Spanning trees

Let $\tau(G)$ denote the set of spanning trees of G. From [3] we know that

$$|\tau(G)| = |S(G)|.$$

This raises the question whether we can give a natural bijection between the two sets, that is invariant under the homomorphism of the graph. However it turns out really fast, that this could not hold in general (see for instance [8, Chapter 4] for more details). This is due to the fact, that there is no group structure on $\tau(G)$, so it is unlikely to have such a bijection. Meanwhile, adding an additional structure on the graph one can give a torsor structure on $\tau(G)$ for $S(G)$, a free, transitive action of $S(G)$ on $\tau(G)$. For this we have to introduce ribbon graphs.

Ribbon graphs

Let G be a graph as above, and $\rho = (\rho_v)_{v \in V(G)}$ an ordering of the edges adjacent to each vertex, in other words ρ_v is a cyclic permutation of $\delta(v) = \{e \in E(G) | v \in e\}$. Sometimes this is also called a *combinatorial embedding of the graph.* One can naturally associate a surface to any ribbon graph in the following way: let us say that a cycle in the graph is ρ-compatible, if all consequent edges in the circle are consequent with respect to ρ. This way, if we thicken the edges of the graph, and gluing disks to all the ρ-compatible cycles we get a surface. Edmonds Jr shows in [6] that these cycles are actually the "faces" of the surface associated to the ribbon graph. Euler's formula then yields the following equation to $g_s(G)$, the *genus of the associated surface*, also referred as the topological genus of the graph:

Proposition 2. *Let (G, ρ) be a ribbon graph, and $cyc(G, \rho)$ denote the number of ρ-compatible cycles. Then*

$$2g_s(G) = 2 - v(G) + e(G) - cyc(G, \rho).$$

Note that this is not the same as the combinatorial genus of the graph, defined as $e(G) - v(G) + 1$.

Bernardi bijection and break divisors

For introducing the action of $S(G)$ on $\tau(G)$, finally we need the following bijection between the spanning trees and $S^g(G)$, where $g = e(G) - v(G) + 1$ is the combinatorial genus.

Let T be a spanning tree. There is $e(G) - (v(G) - 1) = g$ *non-tree edges* in the graph. Putting one chip to either end of non-tree edges yields a configuration in $Div^g(G)$. We call that element b a *break divisor* to T. Naturally, there are various break divisors associated to a tree, depending on which vertex we choose to put a chip on; as well as each break divisor could potentially be associated with many spanning trees. It turns out, that the break divisors give a unique representation of $S^g(G)$ ([1]), so that for every $x \in Div^g(G)$ there is a unique break divisor $b \in Div^g(G)$ (associated to some tree) so that $x \sim b$.

However, it is proved in [2] that one can give an invariant bijection between the spanning trees and the break divisors. This is called the Bernardi bijection. Let us start with an initial spanning tree T. We shall choose a so-called *base vertex* and a *base edge*, adjecent to base vertex. We will walk through the spanning tree, while inspecting the non-tree edges as well, and will put g chips on the vertices, in such a way, that the resulting configuration will be a break divisor of T. The rules are the following: while inspecting the edges of a vertex v

- if we see a non-tree edge, that hasn't been seen before, we put a chip on v and starting to inspect the next edge given by the ordering ρ_v.
- if we see a tree edge, let's say vw, we traverse across it, and start to investigate the vertex w and the following edge with respect to ρ_w.

This naturally yields a break divisor. It is proven that this is actually a bijection, and Baker and Wang gave combinatorial prove of it in [2], by simply giving the inverse of this algorithm. It is important to point out, that this bijection heavily depends on the choice of the basepoints and edges.

Bernardi torsor

Baker and Wang also defined the so called the Bernardi torsor in [2], that goes as follows:

For giving an action $\beta_v : S(G) \times \tau(G) \to \tau(G)$ we shall notice that $S(G)$ naturally acts on $S^g(G)$ with pointwise addition:

$$x \in S(G), b \in S^g(G) : (x, b) \mapsto x + b.$$

Notice that $x + b$ is still an element of $S^g(G)$. This yields the following action of $S(G)$ on $\tau(G)$.

Let us denote the Bernardi bijection from the previous subsection with basepoints v, e as $\Phi_{v,e}$. This way, starting from an initial spanning tree T and $x \in S(G)$ we define the action

$$\begin{aligned} \beta_{v,e} : S(G) \times \tau(G) &\to \tau(G) \\ (x, T) &\mapsto T' : T \xrightarrow{\Phi_{v,e}} b \xrightarrow{S(G)} x + b \xrightarrow{\Phi_{v,e}^{-1}} T'. \end{aligned}$$

That is getting the corresponding break divisor associated to T with respect to v, e, shift the configuration with x, then map it back to get the corresponding spanning tree. Because the break divisors give a unique representation of $S^g(G)$ this is indeed a free, transitive action. Another important thing to point out is that, while the bijection depends on the basepoints v, e it turns out that the Bernardi action only depends on the base vertex only. Due to this fact, sometimes we will only referring to it as β_v, when the base edge is not important.

Topology of ribbon graphs

So far, we introduced a ribbon structure on the graphs, and with the help of it we successfully defined a torsor structure on the spanning trees and and the sandpile group. On the other hand ribbon structures induced a topology on the graph in the form of the associated surface. It turns out that there are actually quite strong connections between the topology of a ribbon graph and the sandpile torsors acting on its spanning trees. One connection, for example is proven in [2] and [5], that if the ribbon graph is planar, that is $g_s(G) = 0$, then the Bernardi action happens to be independent not just only the choice of the base edge, but the base vertex as well. Another interesting coincide is that in the planar case, the Bernardi action and the so called *rotor-routing action*, which gives another torsor for the sandpile group on $\tau(G)$ introduced in [7], actually are the same maps ([2]). This raises the question whether there is an even stronger connection between the topology of a ribbon graph and the sandpile torsors.

It turns out, that there is really. McDonough proved in [9] that we can determine the topological genus of the ribbon graph given the rotor-routing torsor, even if the ribbon structure itself is unknown. That gives a surprising coincide between the topology of a ribbon graph and the sandpile torsors themselves.

In this semester we were trying to prove the same for the Bernardi torsor. We managed to prove it for cut-free ribbon graphs, but proving for arbitrary ribbon graphs is under further examination.

Determine genus from sandpile algorithms

Our main result is the following. For a reminder, we call a graph k-vertex-connected, or k-connected in short, if removing any $k - 1$ vertices from graph leaves the graph connected.

Theorem 3. *Let (G, ρ) be a 3-connected ribbon graph and suppose we are given the maps*

$$\beta_v : S(G) \times \tau(G) \to \tau(G)$$

for all $v \in V(G)$. Then, even if the ribbon structure itself is unknown, we can determine the topological genus $g_s(G)$ of the graph, defined as

$$g_s(G) = 2 - v(G) + e(G) - cyc(G, \rho).$$

At first we will consider 4-connected graphs, because this gives the core of the idea. Extending it to the 3-connected case raises some technicalities only.

Determine genus for 4-vertex connected graphs

Let (G, ρ) be a 4-vertex connected ribbon graph. Take a vertex v and consider three edges adjacent to it, that are in cyclic order (e_1, e_2, e_3) with respect to ρ_v. Let the other vertices adjacent to e_1, e_2, e_3 be w_1, w, w_3 respectively. Call the neighbors of w other than v *lower* neighbors if there exist paths from w_1 to them in $G - \{v, w, w_3\}$, and call them *upper* otherwise, so if all paths from w_1 goes through w_3. In this simpler case, when the graph is 4-connected, there exists no upper neighbor of w, because leaving the vertices v, w and w_3 leaves the graph connected. However, we can choose an arbitrary neighbor of w other than v, let's call it w', and consider the path from w_3 to w' in $G - \{v, w, w_1\}$. Let's call the edge ww' e_w. Note that we can take a spanning tree T that is an extension of the paths described above and which has the following five properties:

- $e_1, e_3 \in T$;
- $e_2 \notin T$;
- $e_w \in T$;
- w_3 is not in any paths in T, going from w_1 to a lower neighbor,;
- w_1 is not in the path in T, going from w_3 to upper neighbor w'.

Our claim is that the Bernardi action with basepoint v, sandpile element

$$x = \begin{cases} x_v = -1 \\ x_w = 1 \\ x_u = 0, u \neq v, w \end{cases}$$

takes the tree T to another spanning tree $T' = T - e_w + e_2$.

Proposition 4. *In the setup described above $\beta_v(x)(T) = T'$ if and only if the cyclic ordering of edges e_1, e_2, e_3 around v is (e_1, e_2, e_3).*

Proof. Let the base edge of the action be e_2. First we prove backwards.

Compare the divisor route associated to T to the one with T'. They will only differ on the edges e_2, e_w. In T, v gets a chip from edge e_2, while in T' the route *goes along*. As the route will immediately turn back from w, the remaining route and therefore divisor will remain the say. This way v looses exactly one chip.

On the other hand, w is reached from w_3, however it was reached from v immediately in T'. The cyclic ordering about v is (e_1, e_2, e_3), hence all the edges connecting the lower neighbors to w were seen from w both in T and T' (because we first routing over the upper tree from e_3, then moving at last to the lower tree from e_1), so w gains only one divisor chip, the one from edge e_w, which was a tree-edge in T, but was first seen from w in T'.

The upper neighbor w' gains no chips other the ones that were gained in T as well. Same happens with the lower neighbors, that were the second seeing the edges adjacent to w. For all other vertices the divisor route remains the same.

For the other direction, suppose that the cyclic ordering around v is (e_2, e_1, e_3). We will show a sandpile element y, that is not equivalent to x in the linear equivalence of $S(G)$, but which takes T to T'. This would indeed imply our claim.

Suppose that there are k lower neighbors of w. Let the sandpile element be

$$y = \begin{cases} y_v = -1 \\ y_w = k+1 \\ y_u = -1, u \text{ is a lower neighbor of } w \\ y_{u'} = 0, \text{ otherwise.} \end{cases}$$

Our claim is that given this this cyclic ordering this takes T to T'.

This is indeed true. With the same reasoning, v looses exactly the edge e_2. However w gains possibly a lot more, because in T' it sees all its neighbors first, while in T the lower neighbors had seen these edges first. Therefore w gains k chips from the lower neighbors and one from e_w, while all lower neighbors looses at least one. It can be easily checked however, that it is exactly one, and as the same way we did before, all other vertices gain no more chips, because the Bernardi route are the same in these cases.

The question is then whether x and y represent the same sandpile element. Suppose they do. Let $s = x - y$ be

$$s = \begin{cases} s_v = 0 \\ s_w = -k \\ s_u = 1, u \text{ is a lower neighbor of } w \\ s_{u'} = 0, \text{ otherwise.} \end{cases}$$

In this case due to Lemma 1 we could reach the all zero configuration via firing and borrowing steps. Because the only way for a vertex to loose chips is via firing, all lower neighbors should fire. This would balance the $-k$ chips on w but would give chips to other, originally zero valued vertices. Then these vertices should fire as well. Therefore we could backtrack until v via w_1. So v reaches at least one chip eventually. Then it should fire, so does w now, as having at least one chip again. Due to the connectedness of the graph all vertices should fire at least once. But firing all vertices is the same as no firing at all, leading to a contradiction. □

The reason why it was worth examining this first case, is because it gives a better understanding on the idea itself, therefore being able now to differentiate between the technicalities and the main idea itself in the next proof.

Determine genus for 3-vertex-connected ribbon graphs

Let (G, ρ) be a 3-connected ribbon graph. Suppose that the cyclic ordering around a vertex v is (e_1, e_2, e_3). Take a similar setup, described above in Proposition 4, except that now $G - \{v, w, w_3\}$ might not be connected. However, we call the neighbors of w, again, lower neighbors, if they are reachable from w_1 in $G - \{v, w, w_3\}$; and upper, if they are not. Note, that due to the connectedness of $G - \{v, w\}$ they are all reachable from w_1 in $G - \{v, w, w_1\}$ as well. Hence, extending these paths to a spanning tree T, we can eventually get a T with the same five properties, as before, that is:

- $e_1, e_3 \in T$;
- $e_2 \notin T$;
- $e_w \in T$;
- w_3 is not in any paths in T, going from w_1 to a lower neighbor,;
- w_1 is not in the path in T, going from w_3 to upper neighbor w'.

Furthermore suppose that

$$\deg(w) = 1 + k + l, \text{ where } k \text{ is the number of upper neighbors, } l \text{ is the number of lower ones.}$$

Label $U = \{a_i\}_{i=1}^k$ the upper and $L = \{b_j\}_{j=1}^l$ the lower neighbors.

Consider the break divisor of T corresponding to Φ_{v,e_2}. At first, vertex v gets a chip from the non-tree edge e_2 then moves forward to e_3. Traversing in the upper neighbors, at some point we will reach e_w therefore w itself. As some of the upper-edges might have already been seen from upper neighbors, but non of the lower-edges, w yields $l + \sum_{i=1}^k \alpha_i$ chips, where the vector α is taken on the upper neighbors and depends on when we saw w comparing to upper neighbors, in other words:

$$\alpha_i = \begin{cases} 1 \Leftrightarrow \text{upper edge } (a_i, w) \text{ corresponds to } a_i \text{ in } \Phi_{v,e_2} \\ 0 \Leftrightarrow \text{upper edge } (a_i, w) \text{ corresponds to } w \text{ in } \Phi_{v,e_2}. \end{cases}$$

In contrary, consider the case when the cyclic order around v is (e_2, e_1, e_3). Take the same spanning tree T. (Note that the construction of T does not rely on the choice of ρ_v.) Similarly consider the break divisor of T corresponding to Φ_{v,e_2}. At first, vertex v gets a chip from the non-tree edge e_2 but then moves forward to e_1! After traversing through the lower neighbors, it moves forward to e_3 and then takes the same route as before. So sometime later it reaches w. All the lower neighbors have already been seen before, hence w only gets $\sum_{i=1}^k \alpha_i$ chips now, where α is the same as before.

Lastly, consider the break divisor of $T' = T - e_w + e_2$ with respect to Φ_{v,e_2}. Now v does not receive a chip from e_2, but moves along, and w sees all its neighbors. All other routes are the same as in the previous cases, except that nor the upper, nor the lower neighbors gets a chip from the edges adjacent to w. This is true in either order of ρ_v.

Consider

$$S_U = \left\{ \big(-1_v, (1 + \sum_{i=1}^{k} \alpha_i)_w, (-\alpha_i)_{a_i}, \underline{0} \big) \,|\, i = 1, \dots, k,\, \alpha \in \{0,1\}^k \right\}$$

and

$$S_L = \left\{ \big(-1_v, (1 + l + \sum_{i=1}^{k} \alpha_i)_w, (-\alpha_i)_{a_i}, (-1)_{b_j}, \underline{0} \big) \,|\, i = 1, \dots, k, j = 1, \dots, l,\, \alpha \in \{0,1\}^k \right\},$$

all subsets of $Div^0(G)$. From the previous explanations the following holds true.

Proposition 5. *In the setup described above*

- *if the cyclic ordering of edges e_1, e_2, e_3 around v is (e_1, e_2, e_3) then there exists an $x \in S_U$ such that $\beta_v(x)(T) = T'$,*
- *while if the cyclic ordering of edges e_1, e_2, e_3 around v is (e_2, e_1, e_3) then there exists a $y \in S_L$ such that $\beta_v(y)(T) = T'$.*

Furthermore, in either case, they share the same α, meaning that

- *If the cyclic order around v is (e_1, e_2, e_3) and $x \in S_U$ is the corresponding element, then if we change te order around v to (e_2, e_1, e_3) then the corresponding $y \in S_L$ has the same vector on the upper neighbors, i.e.: $x|_U = y|_U = \alpha$.*

In order to be able distinguish between the two cases based only on the sandpile torsor, we need the following lemma.

Lemma 6. *The two sets S_U, S_L are sufficiently separable in $S(G)$, meaning that for any $\alpha \in \{0,1\}^k$ on the upper neighbors, the corresponding $x \in S_U, y \in S_L : x|_U = y|_U = \alpha$, are not equivalent: $x \not\sim y$.*

Proof. Suppose there is an $\alpha \in \{0,1\}^k$ that the corresponding two elements x, y are equivalent, so that $s = x - y$ is equivalent with the all zero configuration. However with the same reasoning, this would lead to a necessary firing of all vertices, therefore leading to a contradiction.

Indeed. As

$$s = \big(0_v, -l_w, \underline{0}_U, \underline{1}_L, \underline{0}\big),$$

all lower neighbors should fire in order to loose chips. This necessary firing traverses back to v which should then fire too. So w should fire too. Due to the connectivity of the graph, this yields a necessary firing of all vertices, which is equivalent of firing no vertices. □

We are now ready to prove the main theorem.

Proof of Theorem 3. In a generic case, if both v and w has sufficiently large degrees, we can apply our previous results. Considering spanning trees T and T' as described above, we can check which element of $S(G)$ is taking T to T': according to Proposition 5 we will have an element $x \in S_U(G) \cup S_L(G)$. But according to Lemma 6 we can successfully determine by looking at x, which is the correct cyclic order of ρ_v.

In the degenerate cases: if $\deg(w) = 2$, then it does not have lower neighbors, but it is fine; while if $\deg(w) = 1$, then it does not have an effect of the genus, as it only adds a trivial cycle to G whose position in ρ_v is irrelevant [1]. If $\deg(v) \leq 2$, then the cyclic ordering is trivial around v.

This way, we can reconstruct ρ around every vertex up to a point, where the possible ambiguations does not have any effect on the genus of the graph. □

[1] Actually, more is true, see [9, Lemma 18]. In general if we have two ribbon graphs and glue them together by identifying two arbitrary vertices from each, and the cyclic order around that vertex is trivial then the genus of the given graph is the same for all possible trivial orders.

Further considerations

For arbitrary connected graphs we are not done yet. However it is very likely that the following is true.

Conjecture 1. *Let (G, ρ) be a connected ribbon graph and suppose we are given the maps*

$$\beta_v : S(G) \times \tau(G) \to \tau(G)$$

for all $v \in V(G)$. Then, even if the ribbon structure itself is unknown, we can determine the topological genus $g_s(G)$ of the graph, given as $g_s(G) = 2 - v(G) + e(G) - cyc(G, \rho)$.

The hardness of the proof of Conjecture 1 raises from the fact, that even for 2-vertex-connected graphs we might not be able to find a good tree, where we can have an analog of Propositions 5 or 4.

References

[1] Yang An, Matthew Baker, Greg Kuperberg, and Farbod Shokrieh. Canonical representatives for divisor classes on tropical curves and the matrix–tree theorem. In *Forum of Mathematics, Sigma*, volume 2. Cambridge University Press, 2014.

[2] Matthew Baker and Yao Wang. The bernardi process and torsor structures on spanning trees. *International Mathematics Research Notices*, 2018(16):5120–5147, 2018.

[3] Norman L Biggs. Chip-firing and the critical group of a graph. *Journal of Algebraic Combinatorics*, 9(1):25–45, 1999.

[4] Anders Björner, László Lovász, and Peter W Shor. Chip-firing games on graphs. *European Journal of Combinatorics*, 12(4):283–291, 1991.

[5] Melody Chan, Thomas Church, and Joshua A Grochow. Rotor-routing and spanning trees on planar graphs. *International Mathematics Research Notices*, 2015(11):3225–3244, 2015.

[6] John Robert Edmonds Jr. *A combinatorial representation for oriented polyhedral surfaces.* PhD thesis, 1960.

[7] Alexander E Holroyd, Lionel Levine, Karola Mészáros, Yuyal Peres, James Propp, and David B Wilson. Chip-firing and rotor-routing on directed graphs. In *In and out of equilibrium 2*, pages 331–364. Springer, 2008.

[8] Caroline J Klivans. *The mathematics of chip-firing.* CRC Press, 2018.

[9] Alex McDonough. Determining genus from sandpile torsor algorithms. *arXiv preprint arXiv:1804.07807*, 2018.