The dynamics of the Hopfield model for homogeneous weight matrix

BY TAKI EDDINE DJEBBAR

1.INTRODUCTION : The neural network has recently attracted many mathematicians and one of the simplest model is called Hopfield model ,it defines as a reccurent neurel network ,the main aim in this paper is to determine the dynamics of this model in special structure of the weight matrix but first let us introduce the differntial equations which discribed by hopfield and what the weight matrix is :

$\dot{x} = Dx + Wy + I \quad , \quad y_i = f(x_i)$..........................(*)

where x denotes the membrane potential and y represents the firing rate of neurons , I is the external input, the matrix W contains the strengths of connections and f is the activation function. in this paper we will introduce some types of the so called bifurcations and the steady states of the model, the bifurcation which we present in this paper we calculate using analytical and numerical tools .in special a structure of W

$$w = \begin{bmatrix} 0 & w_2 & \cdots & w_n \\ w_1 & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & w_n \\ w_1 & w_2 & \cdots & 0 \end{bmatrix}$$

We use the sigmoid function $f(x) = \frac{1}{(1+e^{(\mathrm{a}-\mathrm{bx})})}$ as activation function ,b > 0 and let D be identity matrix and assume that the neurons do not receive any external input ,then the model of system in the top can be written as $\dot{x}_i = -x_i + \sum_{k=1}^{n} w_k f(x_k) - w_i f(x_i)$................(1)

2. The homogenous network : if the network contains neurons with equal weights then the system (2) can be traced back to a lower dimensional system so we have the theorem which says : If i and j such indices that $w_i = w_j > 0$, then t → $|x_i(t) - x_j(t)|$ is strictly

decreasing and lim t→+∞ $|x_i(t) - x_j(t)|$ = 0, i.e. x_i equals x_j in steady states and on periodic orbits.

let us investigate the dynamics of the Hopfield model when the size of the network is arbitrary but each neuron has the same w > 0 weight. According to previous Theorem , the solutions become asymptotically equal so it is enough to study the behavior of a one-dimensional system instead of system (1).

Corollary : If there exists such w that $w = w_i \; \forall$ i, then the

$$x^{\cdot} = -x + (n - 1)wf(x) \qquad (2)$$

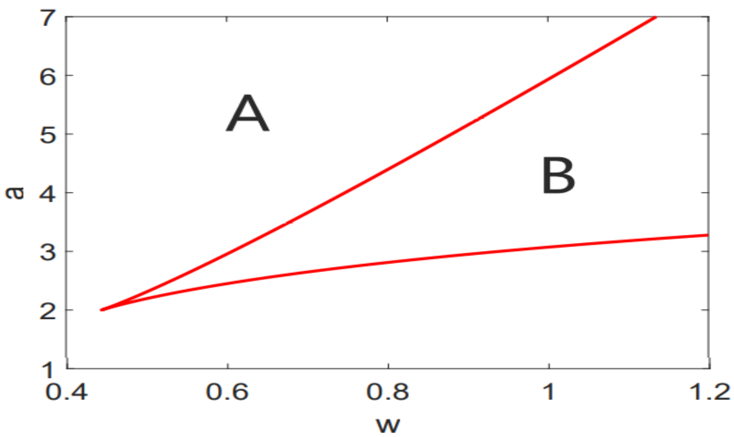
determines the asymptotic behaviour of system (1).

We determine by the saddle-node bifurcation curve the bihaviour of the system. First, we look at the equilibrium points (RHS(2)=0): $-x + (n - 1)wf(x) = 0$ (3)

We compute the derivative of the equation : $-1 + (n - 1)wf'(x) = 0$ (4)

Let us apply the activation function $f(x) = \frac{1}{(1 + e^{(a - bx)})}$ for (3) and (4) and get the bifurcation parameter w then substitute it into (4) and we get the saddle-node bifurcation curve parametrized with x .

We fix b=1 and the neuros n=10 to plot the bifurcation curve The saddle-node curve divides the parameter plane into two domains A and B (A has one equilibruim point which is globaly stable) (B has 3 equilibruim points Two of the three steady states are stable while the third one is unstable) as we have seen in this figure



<u>3.the effect of an inhibitory neurons :</u> we consider a network contains n-1 neurons with wight w >0 and one neuron has arbitrary wight w_1 arbitrary so one column of the weight matrix W is different from the others,we choose the weights w and w_1 as bifurcation parameters then determine the dynamics of the model ,if there exists $w_i > 0 \;\; \forall$ i=2,n according to theorem 1 then the solutions become asymptotically equal (x_1, x_2) satisfies system (2), where $x_2 = x_i \; \forall$ i = 2, ..., n and the weight w_1 of x_1 is arbitrary.

Corollary :if there exists $w=w_i >0 \; \forall$ i = 2, ..., n and w_1 arbitrary then we have the differential system describe the asymptotic behavior of system (1):

$$x_1^{\cdot} = -x_1 + (n-1)wf(x_j) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$x_j^{\cdot} = -x_j + (n-2)wf(x_j) + wf(x_1) \quad ....(6)$$

3.1 Saddle-node bifurcation: we search about which parameter values cause the change in the number of steady states, we determine the equilibrium points from (5) and (6) and eliminate x_1 from (5) we plug x_2 in (6) then we get :

$$-x_2 + w_1 f((n-1)wf(x_2)) + (n-2)wf(x_2) = 0 \quad ....(7)$$

Then we derivate it and get w_1 from (7) and sustitute it in the derivative then we get the equation

$$-1 + \frac{x_2 - (n-2)wf(x_2)}{f((n-1)wf(x_2))} f'((n-1)wf(x_2))(n-1)wf'(x_2) + (n-2)wf'(x_2) = 0 \quad ....(8)$$

This equation determines the saddle-node bifurcation curve in system (5) and (6) .we substitute w from derivative of (7) and and approximate it numerically using bisection method .

3.2 Andronov-Hopf bifurcation: let us investigate their stability of (5) and (6) by the equilibrium points . we determine the Jacobian matrix of system (5)-(6):

$$J = \begin{pmatrix} -1 & (n-1)wf'(x_2) \\ f'((n-1)wf(x_2)) & -1 + (n-2)wf'(x_2) \end{pmatrix}$$

We apply the necessary condition of the bifurcation : I) trace(J)=0 II) det(J)>0

we use that the activation function $f(x) = \frac{1}{(1+e^{(a-bx)})}$ satisfies $f' = bf(1-f)$

we eliminate w from $-2 + (n-2)wf(x_2) = 0$ and we have w_1 from the derivative of

(7) so we get the folowing : $w_1 = \frac{x_2 - (n-2)wf\prime(x_2)}{f'((n-1)wf(x_2))}$

$$w = \frac{2}{(n-2)f(x_2)(1-f(x_2))b}$$

$$0 < 1 - (n-2)wbf(x_2)(1-f(x_2)) - (n-1)ww_1 b^2 f(x_2)(1-f(x_2)) f((n-1)wf(x_2)) \left(1 - f((n-1)wf(x_2))\right).$$

These parameters give the Andronov-Hopf bifurcation curve.

3.5 Dynamics of the Hopfield model: we fix the parameters a = 2, b = 1 and n = 10 to plot all bifurcation curves we have determined ,there are 3 domains a domain A of the system (5)-(6) has one equilibrium which is globally stable Crossing the saddle-node bifurcation curve we can find three steady states in domain B. The first one is unstable, the second one is a saddle and the third one is stable. The first fixed point becomes stable if we cross the Andronov-Hopf bifurcation curve and an unstable limit cycle appears around it in the domain C.

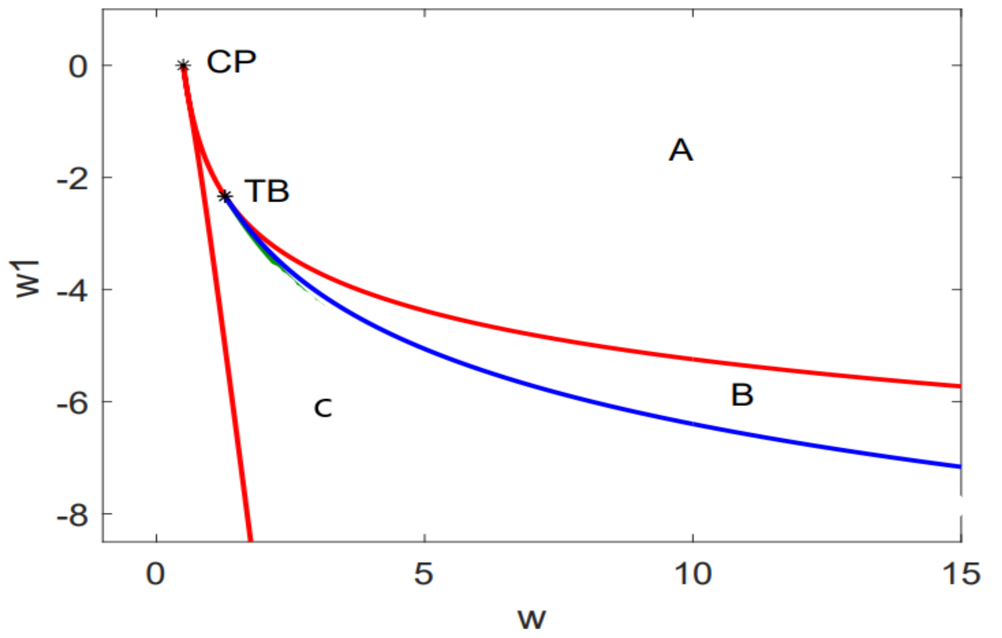


Figure 2: Saddle-node (red), Andronov-Hopf (blue) bifurcation curves in system (5)-(6) with a = 2, b = 1 and n=10

4.Conclusion: The main goal of this paper is to determine the dynamics of the Hopfield model given in (*) we studied the model with n neurons and have the same positive wight we call that the homogenous network we find how the bihavour of model's system and we get the saddle node bifurcations and we use a and w as bifurcations parameters then we consider a special network which is inhibitory network we take n-1 neuron with the same positive wight and one neuron with arbitrary wight w_1 and we use w and w_1 as bifurcation parameters then we get the number of steady states of the system we use the saddle node bifurcation to investigate the steady states then we use andronov-hopf bifurcation to know the stability of the equilibrium points then We determined the behaviour of the model in each domain with fixed parameters a = 2, b = 1 and n = 10 then we matlab to show the steady states of the equilibrium and their stability and phase portrais of the model (in presentation).

References :

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