MODELLING PROJECT REPORT

COST-OPTIMAL, CONDITION-BASED MAINTENANCE

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Introduction

In order to prevent catastrophic failure of a system and minimize the costs of repair, maintenance polices are arranged for systems that are subject to deterioration. In many cases, it is not known if an object is deteriorating or not without inspection, which is organized aiming for early detection of deterioration. However, performing a large number of inspections could be costly and unnecessary. Therefore, statistical modelling and inference techniques are used in order to find appropriate maintenance policies.

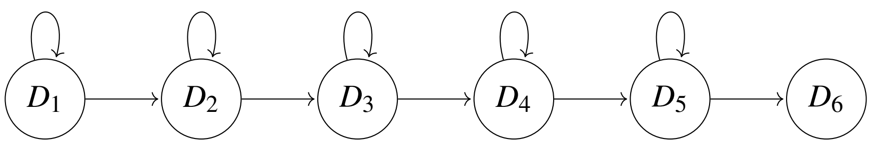
Markov chain based-models are powerful tools for simulating degradation and optimizing maintenance policies [9]. It is has been significantly used in the medical field to model disease progression and optimize screening, the theory of periodic screening was introduced by [10] and later extended by many researchers [7, 8]. Markov chain models are also used in optimizing maintenance programs in the wind industry [2] and in transportation e.g. modelling the deterioration of highway infrastructures[3], and bridge management system [4].

In probabilistic terms, Markov chain based-models describe the dynamic behaviour of a system over time, where the states of the system (the condition) describe the level of accumulated damage and the transition probability matrix describes the dynamics of the deterioration. The main assumption in a Markov chain model is that the information of the current state in the process is sufficient to describe the future probabilistic behaviour of the process.

In this modeling project, we present a continuous-time Markov chain as a simplified probabilistic model to deal with accumulated damage that can be described by a discrete number of states. We assume that we have six states of deterioration, and we aim to provide a cost-optimal maintenance strategy. The model can be used in many application, e.g. disease progressions such as cancer, insurance claims and cost-optimal inspection policies.

Model

Consider a continuous-time Markov chain $(X_t)_{t\geq 0}$ in a space I consisting of six states D_1, D_2, D_3, D_4, D_5 and D_6 describing the level of damage, where D_1 is the damage-free state, $D_2, \ldots, D_5$ describe the level of damage and D_6 is the symptomatic state where the degradation is exposed by showing symptoms. Suppose that the propagation between the states is governed by an exponential waiting time in each state. Suppose that the progression starts from the damage-free state D_1 and progresses into the next states. We chose to use five states to mimic cancer progression models, where D_1 can be thought of as the disease-free state and $D_2, \ldots, D_5$ are the cancer stages at diagnosis I-IV [1]. The aim of screening is to detect the damage as early as possible in hopes of improving survival.
Let us start laying the foundation of the model, let Y_i be the waiting time at state D_i before deteriorating to the next state, and suppose that $Y_i \sim exp(\lambda_i)$ for $i = 1, 2, \ldots, 5$.



Furthermore, suppose that an inspection program is organized starting from $\tau_0 = 0$ and $\tau_i = \tau_1 + (i-1)\Delta$, where τ_i is the age of an item at the i^{th} inspection, and Δ is the inter-inspection time. We assume that the inspections may have a false positive rate, which is the probability of falsely detecting deterioration, i.e. the item is in the damage-free state D_1 but the inspections falsely show that it is in a deteriorating state. Also, we assume perfect inspection sensitivity, which means inspections will detect deterioration with probability 1. The aim is to derive the cost-optimal periodic inspection strategy.

The expected total cost $E(TC(\tau_1,\Delta))$ in this setup is divided into four parts:

- The expected cost of repair screened items $E(CR(\tau_j,\Delta))$.
- The expected cost of repair symptomatic cases $E(CS(\tau_j,\Delta))$.
- The expected cost of inspections $E(CI(\tau_j,\Delta))$.
- The expected cost associated with identifying false positives $E(CFP(\tau_j,\Delta))$.

In order to calculate the expected cost of repair, we need to derive the distribution of degradation at τ_j for $j = 1,2,\dots,K$, denoted by X_{τ_j}, where K is the total number of inspections in an observation period of length T, that is given by $K = \left\lceil \frac{T-\tau_1}{\Delta} \right\rceil$ if $\tau_1 \leq T$ and 0 otherwise. The density of the convolution of waiting times Y_i is straightforward to compute [5], and given for $\lambda_i \neq \lambda_j$:

$$f_{Y_1+Y_2+\cdots+Y_i}(y) = \left[\prod_{k=1}^{i} \lambda_k\right] \sum_{j=1}^{i} \frac{e^{-\lambda_j y}}{\prod_{\substack{k\neq j\\ k=1}}^{i}(\lambda_k-\lambda_j)}, \quad y>0,\ i\geq 1.$$

In case there are identical parameters, the distribution of the sum of the random variables $Y_1,Y_2,\dots,Y_r$ was established by H. Jasiulewicz and W. Kordecki [6] as follows

$$f_{Y_1+\cdots+Y_r}(t) = \sum_{i=1}^{n} \lambda_i^{k_i} e^{-t\lambda_i} \sum_{j=1}^{k_i} \frac{(-1)^{k_i-j}}{(j-1)!} t^{j-1} \times \sum_{\substack{n_1+\cdots+n_n=k_i-j\\ n_i=0}} \prod_{\substack{l=1\\ l\neq i}} \binom{k_l+n_l-1}{n_l} \frac{\lambda_l^{k_l}}{(\lambda_l-\lambda_i)^{k_l+n_l}} \mathbb{1}_{t>0},$$

where $\lambda_1,\dots,\lambda_n$ are distinct parameters and k_i denote the number of components with the same parameter λ_i.

In this report, we will assume that the first scenario where all parameters are different. As a first step, we will find an cost optimal inspection program assuming that when a deterioration is detected, a repair is performed and the item leaves the chain (i.e. a repaired item will not be inspected again). In other words, we will minimize the expected costs for a single cycle, which lasts from the disease free state till repair.

Formulas for the probabilities

Denote by $Q_{Y_i}(t) = \int_t^\infty f_{Y_i}(x)dx$ is the survivor function of the waiting time Y_i.

Proposition 1. *The distribution of the degradation at the first inspection X_{τ_1} is given by:*

$$P(X_{\tau_1}=D_i)=\begin{cases} e^{-\lambda_1\,\tau_1} & i=1 \\ \displaystyle\prod_{k=1}^{i-1}\lambda_k\sum_{k=2}^{i}\frac{e^{-\lambda_k\tau_1}-e^{-\lambda_1\tau_1}}{\displaystyle\prod_{\substack{l=1\\ l\neq k}}^{i}(\lambda_l-\lambda_k)} & i\in\{2,3,4,5\} \end{cases}$$

Proposition 2. *The distribution of the degradation at the j^{th} inspection X_{τ_j} is given by:*

$$P(X_{\tau_j}=D_i)=\begin{cases} e^{-\lambda_1\,\tau_j} & i=1 \\ \displaystyle\prod_{k=1}^{i-1}\lambda_k e^{-\lambda_1\tau_{j-1}}\sum_{k=2}^{i}\frac{e^{-\lambda_k\Delta}-e^{-\lambda_1\Delta}}{\displaystyle\prod_{\substack{l\neq k\\ l=1}}^{i}(\lambda_l-\lambda_k)} & i\in\{2,3,4,5\} \end{cases}$$

Proposition 3. *The probability of showing symptoms before the first inspection τ_1, denoted by $I(\tau_1)$, is:*

$$I(\tau_1)=\prod_{i=1}^{5}\lambda_i\sum_{j=1}^{5}\frac{1-e^{-\lambda_j\tau_1}}{\displaystyle\prod_{\substack{k=1\\ k\neq j}}^{5}\lambda_j(\lambda_k-\lambda_j)}$$

Similarly, the probability of showing symptoms between the last inspection τ_K and the end of the observation period T, denoted by $I(\tau_{K+1})$, is:

$$I(\tau_{K+1})=e^{-\lambda_1\tau_K}-e^{-\lambda_1 T}.$$

Proposition 4. *The probability of showing symptoms between τ_{j-1} and τ_j for $j=2,\dots,K$, denoted by $I(\tau_j)$, is:*

$$\begin{aligned} I(\tau_j)=&e^{-\lambda_1\tau_{j-1}}-e^{-\lambda_1\tau_j}+\prod_{i=1}^{4}\lambda_i\sum_{k=2}^{4}\frac{e^{-\lambda_k\tau_j}\left(e^{(\lambda_k-\lambda_1)\tau_j}-e^{(\lambda_k-\lambda_1)\tau_{j-1}}\right)}{\lambda_k\displaystyle\prod_{\substack{l\neq k\\ l=1}}^{4}(\lambda_l-\lambda_k)} \\ &+\sum_{k=3}^{5}\frac{\displaystyle\prod_{i=1}^{4}\lambda_i}{\displaystyle\prod_{\substack{l\neq k\\ l=2}}^{5}(\lambda_l-\lambda_k)}\frac{e^{-\lambda_2\tau_j}\left(e^{(\lambda_2-\lambda_1)\tau_j}-e^{(\lambda_2-\lambda_1)\tau_{j-1}}\right)}{\lambda_2-\lambda_1}+\prod_{i=1}^{4}\lambda_i\sum_{k=3}^{5}\frac{e^{-\lambda_k\tau_j}\left(e^{(\lambda_k-\lambda_1)\tau_j}-e^{(\lambda_k-\lambda_1)\tau_{j-1}}\right)}{(\lambda_1-\lambda_k)\displaystyle\prod_{\substack{l\neq k\\ l=2}}^{5}(\lambda_l-\lambda_k)} \end{aligned}$$

Expected costs

Suppose that the cost of repair is an increasing function $CR: I\to\mathbb{R}^+$, where $CR(i)$ is the cost of repair at damage level D_i for $i=1,2,\dots,6$, then the expected cost of repair of damage items which are detected at inspection τ_j is:

$$E(CR(\tau_j,\Delta)) = \sum_{i=1}^{5} CR(i)\cdot P(X_{\tau_j} = D_i).$$

Denote by C_S the cost of repair items which are detected by showing symptoms (i.e. reaching the state D_6), then the expected cost of repair of cases showing symptoms between τ_{j-1} and τ_j is:

$$E(CS(\tau_j,\Delta)) = CR(6)\cdot I(\tau_j).$$

The expected cost associated with identifying false positive is derived using: the probability of being in the deterioration-free state at τ_j, the false positive rate α and the cost of identifying a single false positive C_{FP}. Namely:

$$E(CFP(\tau_j,\Delta)) = C_{FP}\cdot\alpha\cdot P(X_{\tau_j} = D_1).$$

In order to determine the expected cost of inspection, we assume that a repaired item will not be inspected again. Then, the cost of inspection in such a single cycle follows from the number of items participating in each inspection. Suppose that C_I is the cost of inspection for a single item, then the expected cost of the inspection occurring at τ_j is:

$$E(CI(\tau_j,\Delta)) = C_I\cdot P_{\tau_j}(S),$$

such that $P_{\tau_1}(S) = 1 - I(\tau_1)$ and $P_{\tau_j}(S)$ is the probability of an item participating in inspection τ_j;

$$P_{\tau_j}(S) = P_{\tau_1}(S) - \sum_{i=1}^{j-1} H(\tau_i) - \sum_{i=2}^{j} I(\tau_i), \text{ for } \quad j = 2,\ldots,K,$$

where $H(\tau_i) = I(\tau_i) + \sum_{k=1}^{i} P_{\tau_k}(S)$ is the probability of a deterioration getting detected in inspection τ_i.

Putting everything together, the expected total cost for a single cycle is a function of the first inspection and the inter-inspection time and is given by:

$$\begin{aligned} E(TC(\tau_1,\Delta)) &= \sum_{j=1}^{K+1} E(CS(\tau_j,\Delta)) + \sum_{j=1}^{K}\left[E(CR(\tau_j,\Delta)) + E(CFP(\tau_j,\Delta)) + E(CI(\tau_j,\Delta))\right] \\ &= \sum_{j=1}^{K+1} C_s\cdot I(\tau_j) + \sum_{j=1}^{K}\left[\sum_{i=1}^{5} CR(i)\cdot P(X_{\tau_j} = D_i) + C_{FP}\cdot\alpha\cdot P(X_{\tau_j} = D_1) + C_I\cdot P_{\tau_j}(S)\right]. \end{aligned}$$

Simulation and results

Using the following parameterizations, we implement the model in the statistical software R.

	Parameter and Value
Waiting times (in years)	$1/\lambda_1 = 50,\ \ 1/\lambda_2 = 3.3,\ \ 1/\lambda_3 = 2.5,\ \ 1/\lambda_4 = 2.2,\ \ 1/\lambda_5 = 2$
Costs of Repair	$CR(1) = 0,\ \ CR(2) = 17000,\ \ CR(3) = 18500,\ \ CR(4) = 19900,\ \ CR(5) = 21000$
Other costs	$C_{FP} = 550,\ \ C_S = 26000,\ \ C_I = 300$
Observation period (in years)	$T = 100$.
False positive rate	$\alpha = 0.05$

Using the *ggplot* package in R, we ploted the following contour plots:

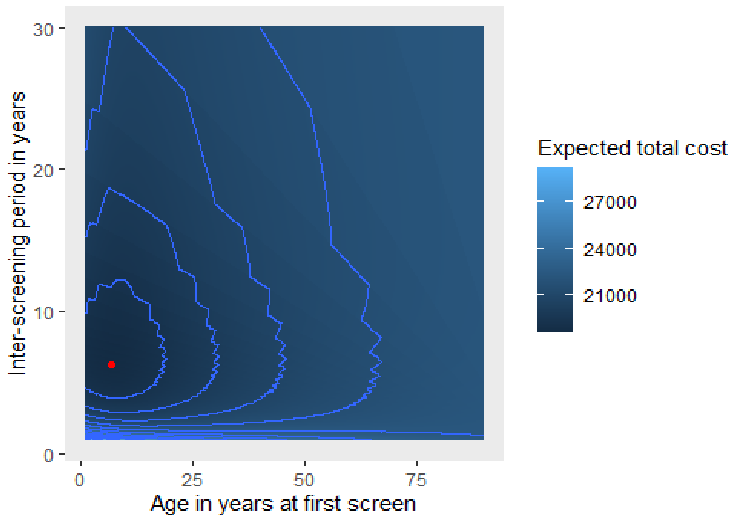


Figure 1: Contour plots for the total expected cost as a function of first inspections and inter-inspection periods. The minimum is represented by the red dot.

Current work and aims

We are currently working on generalizing the model so that we allow recurrence. That is, whenever an item is repaired, it goes back to the damage-free state immediately. In this scenario, we are aiming to find a closed form of the expected costs, or derive a simple method to give a compact upper bound of the expected costs. We also aim to release the perfect sensitivity assumption, that means that there will be a positive probability that the damage will not be detected by an inspection. Another possible generalization is releasing the perfect repair assumption, meaning that an item repaired at damage level D_i will no longer immediately move back to the disease free state D_1, but will move to state D_j where $j < i$. These generalizations will render the model applicable in many scenarios.

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