Cost-optimal, condition-based maintenance

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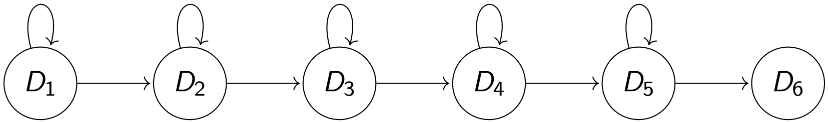
Introduction

- Consider a continuous-time Markov chain $(X_t)_{t\geq 0}$ in a space I consisting of six states $D_1, \ldots, D_6$ describing the level of damage.
- let $Y_i \sim exp(\lambda_i)$ for $i = 1, 2, \ldots, 5$ be the waiting time at state D_i.
- Assume that the inspection is periodic with an inter-inspection time Δ and denote by $\tau_i = \tau_1 + (i-1)\Delta$ the age at the i^{th} inspection.
- Assume that the inspections may have a false positive rate and perfect sensitivity.
- Assume that the damage can only be detected either by an inspection or by showing symptoms (reaching state D_6).
- Within such a setup, our aim is to establish a cost-optimal inspection program.



Expected costs

In this setup, the expected total cost $E(TC(\tau_1, \Delta))$ can be divided into four parts:

- The expected cost of repair screened items $E(CR(\tau_1, \Delta))$.
- The expected cost of repair symptomatic cases $E(CS(\tau_1, \Delta))$.
- The expected cost of inspections $E(CI(\tau_1, \Delta))$.
- The expected cost associated with identifying false positives $E(CFP(\tau_1, \Delta))$.

Our assumptions are:

- Suppose that the cost of repair is an increasing function $C_r : I \to \mathbb{R}^+$.
- Suppose that we have a constant false positive rate α and denote the cost of identifying a single false positive C_{FP}.
- Denote by C_I the cost of inspection for a single item.

Distribution of X_{τ_j} on inspections

- The first step is to determine the distribution of X_{τ_j} on each inspection for $j = 1, \ldots, K$ where $K = \left\lceil \frac{T - \tau_1}{\Delta} \right\rceil$.
- The distribution of X_{τ_j} is computed using the convolution of waiting times Y_i and for distinct parameters (λ_k), that is [1] :

$$f_{Y_1+Y_2+\cdots+Y_i}(y) = \left[\prod_{k=1}^{i} \lambda_k\right] \sum_{j=1}^{i} \frac{e^{-\lambda_j y}}{\prod_{\substack{k \neq j \\ k=1}}^{i} (\lambda_k - \lambda_j)}, \quad y > 0, \;\; i \geq 1.$$

- In case the λ_k's are not distinct, the distribution of the sum of waiting times can also be computed [3]. We assume that $\lambda_i \neq \lambda_j \;\; \forall i, j \in I$.

Distribution of X_{τ_j} on inspections II

Using the convolution of the waiting times, we derived the distribution of the degradation at the j^{th} inspection X_{τ_j}, where $j \geq 2$, as:

$$P(X_{\tau_j} = D_i) = \begin{cases} e^{-\lambda_1 \, \tau_j} & i = 1 \\ \prod_{k=1}^{i-1} \lambda_k e^{-\lambda_1 \tau_{j-1}} \sum_{k=2}^{i} \frac{e^{-\lambda_k \Delta} - e^{-\lambda_1 \Delta}}{\prod_{\substack{l \neq k \\ l=1}}^{i} (\lambda_l - \lambda_k)} & i \in \{2,3,4,5\} \end{cases}$$

Then, the expected cost of repair of damaged items detected by inspection is:

$$E(CR(\tau_1, \Delta)) = \sum_{i=1}^{5} C_r(i) \cdot P(X_{\tau_j} = D_i)$$

Expected repair cost of items with symptoms

The second part of the expected total costs comes from the repair of symptomatic items i.e. items which reach D_6. Hence:

- The probability of showing symptoms before the first inspection τ_1, denoted by $I(\tau_1)$, is:

$$I(\tau_1) = \prod_{i=1}^{5} \lambda_i \sum_{j=1}^{5} \frac{1 - e^{-\lambda_j \tau_1}}{\prod_{\substack{k=1 \\ k \neq j}}^{5} \lambda_j (\lambda_k - \lambda_j)}$$

- The probability of showing symptoms between the last inspection τ_K and the end of the observation period T, denoted by $I(\tau_{K+1})$, is:

$$I(\tau_{K+1}) = e^{-\lambda_1 \tau_K} - e^{-\lambda_1 T}.$$

Expected repair cost of items with symptoms II

- The probability of showing symptoms between τ_{i-1} and τ_i for $i = 2, \dots, K$ denoted by $I(\tau_i)$ is computed in a similar manner.

Using these probabilities, the expected cost of repair of items showing symptoms between τ_{j-1} and τ_j is:

$$E(CS(\tau_1, \Delta)) = \sum_{j=1}^{K+1} C_r(6) \cdot I(\tau_j).$$

Note that this includes the costs of repair before the first inspection and after the last inspection till the end of the observation period.

Expected cost of identifying false positive items

The expected cost of identifying false positive items is derived using:

- The probability of being in the damage-free state at τ_j, denoted by $P(X_{\tau_j} = D_1)$.
- The false positive rate α and the cost of identifying a single false positive C_{FP}.

The probability of an inspection to falsely detect damage when there is none is simply $\alpha \cdot P(X_{\tau_j} = D_1)$. Hence, the expected cost is:

$$E(CFP(\tau_1, \Delta)) = C_{FP} \cdot \alpha \cdot P(X_{\tau_j} = D_1).$$

Expected cost of inspections

- We assume that a repaired item will not be inspected again.
- Denote by $P_{\tau_j}(S)$ the probability of an item participating in inspection τ_j, then $P_{\tau_1}(S) = 1 - I(\tau_1)$ and

$$P_{\tau_j}(S) = P_{\tau_1}(S) - \sum_{i=1}^{j} I(\tau_i) + \sum_{i=1}^{j}\sum_{s=2}^{6} P(X_{\tau_i} = D_s), \text{ for } \quad j = 2, \ldots, K,$$

Therefore, the expected cost of the inspection occurring at τ_j is:

$$E(CI(\tau_1, \Delta)) = C_I \cdot P_{\tau_j}(S).$$

Summing up

Putting everything together, the expected total cost for a single cycle is a function of the first inspection τ_1 and the inter-inspection time Δ and is given by:

$$
\begin{aligned}
E(TC(\tau_1,\Delta)) &= \sum_{j=1}^{K+1} E(CS(\tau_1,\Delta)) + \sum_{j=1}^{K} \big[E(CR(\tau_1,\Delta)) + E(CFP(\tau_1,\Delta)) \\
&\quad + E(CI(\tau_1,\Delta))\big] \\
&= \sum_{j=1}^{K+1} C_r(6)\cdot I(\tau_j) + \sum_{j=1}^{K} \Big[\sum_{i=1}^{5} C_r(i)\cdot P(X_{\tau_j} = D_i) \\
&\quad + C_{FP}\cdot\alpha\cdot P(X_{\tau_j} = D_1) + C_I\cdot P_{\tau_j}(S)\Big].
\end{aligned}
$$

The expected total cost will be minimized using nonlinear minimization (*nlm*) in order to find optimal τ_1 and Δ.

Simulation and results

Using the following parameterizations, we implement the model in the statistical software R.

	Parameter and Value
Waiting times (in years)	$1/\lambda_1 = 50,\ 1/\lambda_2 = 3.3,\ 1/\lambda_3 = 2.5,\ 1/\lambda_4 = 2.2,\ 1/\lambda_5 = 2$
Costs of Repair	$CR(1) = 0,\ CR(2) = 17000,\ CR(3) = 18500,\ CR(4) = 19900,\ CR(5) = 21000$
Other costs	$C_{FP} = 550,\ C_S = 26000,\ C_I = 300$
Observation period (in years)	$T = 100$.
False positive rate	$\alpha = 0.05$

Plot

Using the *ggplot* package in R, we ploted the following contour plot:

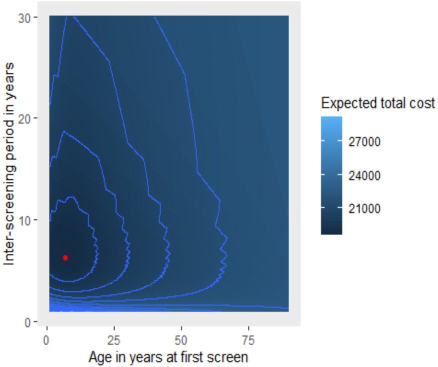


Figure: Contour plots for the total expected cost as a function of first inspections and inter-inspection periods. The minimum is represented by the red dot.

Current work and aims

- We are currently working on generalizing the model so that we allow recurrence. In this scenario, we are aiming to find a closed form of the expected costs, or derive a simple method to give a compact upper bound of the expected costs.
- We also aim to release the perfect sensitivity and the perfect repair assumptions.
- These generalizations will render the model applicable in many scenarios.

References

[1] Markus Bibinger. Notes on the sum and maximum of independent exponentially distributed random variables with different scale parameters. *arXiv preprint arXiv:1307.3945*, 2013.

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[4] Yi Yang and John Dalsgaard Sørensen. Cost-optimal maintenance planning for defects on wind turbine blades. *Energies*, 12(6):998, 2019.

THANK YOU FOR LISTENING