Chromatic Number of the Delaunay Graph with Respect to Axis-Parallel Rectangles

Zoltán Molnár-Sáska

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1 Introduction

We are given a finite point set P of size n in the plane in general position (no two points have the same x-coordinate or y-coordinate). We define the *Delaunay graph* $D(P)$ as the graph on vertex set P where two points $p, q \in P$ are connected by an edge if and only if the smallest axis-parallel rectangle with corners at p and q contains no other point of P. The study of $\chi(D(P))$ was proposed by Chen, Pach, Szegedy and Tardos [1], after Even, Lotker, Ron and Smorodinsky [2] posed the following question arising from frequency assignment in cellular networks: does there exist a constant $c > 0$ such that $\alpha(D(P)) \geq cn$ for every point set P? Chen et al. answered this negatively by proving that $\chi(D(P))$ is unbounded. The goal of this project is to explore the bounds on this chromatic number $\chi(D(P))$ further and review the literature on this problem.

The best known bounds are the following.

Theorem 1.1 (Chen-Pach-Szegedy-Tardos [1], Chan [3], Jin-Kwan-Lichev [4])**.** *For any set P of n points in general position,*

$$\Omega\left(\frac{\log n}{\log \log n}\right) \;\leq\; \chi(D(P)) \;\leq\; O(n^{0.368}).$$

As we can see, the gap is extremely large, so there is still space to improve on both sides. It is believed by many that the upper bound will also turn out to be polylogarithmic, but this problem is still open. We also note that it is hard to give explicit constructions just to even show that the chromatic number is unbounded, Damásdi gave one recently in [5].

In this report, first we will introduce some definitions, then review the upper and lower bounds in separate sections, and finally discuss some programmatic search for constructions, speculative ideas and related problems.

2 Definitions and Notation

Throughout this report, G denotes a graph on n vertices, $\chi(G)$ its *chromatic number* and $\alpha(G)$ its *independence number*. A basic and frequently used fact is that

$$\chi(G) \;\geq\; \frac{n}{\alpha(G)} \tag{1}$$

since any proper coloring partitions the vertices into $\chi(G)$ independent sets, each of size at most $\alpha(G)$.

We define the *dominance order* on P by $p \preceq q$ iff $p_x \leq q_x$ and $p_y \leq q_y$, and the *anti-dominance order* by $p \preceq' q$ iff $p_x \leq q_x$ and $p_y \geq q_y$. We say p *covers* q if $p \prec q$ and there is no $r \in P$ with $p \prec r \prec q$. The *Hasse diagram* of a poset is the graph on its elements where two points are adjacent iff one covers the other. We denote by H_1 and H_2 the Hasse diagrams of the dominance and anti-dominance orders on P respectively. Note that H_1 and H_2 are triangle-free, and that $D(P) = H_1 \cup H_2$.

3 Upper Bounds

3.1 A first bound using the Erdős-Szekeres Theorem

Theorem 3.1. *Any set of n points admits a proper coloring of $D(P)$ using $O(\sqrt{n})$ colors.*

Proof. We can solve this recursively. We sort the points of P by x-coordinate and apply Erdős-Szekeres on the sequence of y-coordinates to find a monotone subsequence of length $\sqrt{n}$. We color every second point of this subsequence with the same new color and then recursively solve the problem for the remaining points. Clearly, this coloring will be proper, as any edge of $D(P)$ that has one endpoint in the newly colored subsequence must have the other endpoint outside of it. Now to count the number of colors used, we can just solve the following recurrence:

$$f(n) \leq 1 + f(n - \lfloor \sqrt{n}/2 \rfloor),$$

which gives $f(n) = O(\sqrt{n})$. □

This idea is simple, but it will be used in other results too that give stronger bounds.

3.2 Conflict-Free Coloring

We start with a definition, we say that a coloring of P is a *conflict-free coloring* if every nonempty axis-parallel rectangle contains a point whose color is unique among all points of P in that rectangle.

Clearly, this is a stronger property than a proper coloring, meaning that every conflict-free coloring of P is a proper coloring of $D(P)$. Interestingly, all the bounds we explore next are proved for conflict-free coloring, so they hold in particular for proper coloring of $D(P)$ as well. Har-Peled and Smorodinsky [6] proved that $O(\sqrt{n})$ colors are enough even for conflict-free coloring, so the bound of Theorem 3.1 holds in this version as well. From now on we consider conflict-free colorings for the improved bounds.

3.3 Improving to $O(n^{6/13})$

Although this is not the best known upper bound, this version shows the main ideas that go into all the improvements. We will combine the Erdős-Szekeres idea with a grid decomposition and *quasi-conflict-free colorings.*

Given a pointset P and a grid G_r partitioning the plane into $r \times r$ cells, a coloring of P is *quasi-conflict-free* (QCF) with respect to G_r if every axis-parallel rectangle contained entirely within a single row or column of G_r is conflict-free.

There is a lemma shown in [7] which we will just use as a black box.

Lemma 3.2 (Elbassioni-Mustafa [7])**.** *If each row and column of G_r contains at most B points of P, then P admits a QCF coloring with respect to G_r using $\tilde{O}(B^{3/4})$ colors.*

(we write $\tilde{O}$ here to hide polylogarithmic factors in B).

Now we just proceed to show the following:

Theorem 3.3 (Ajwani-Elbassioni-Govindarajan-Ray [8])**.** *Any set of n points admits a conflict-free coloring with respect to axis-parallel rectangles using $\tilde{O}(n^{6/13})$ colors.*

Proof. Let us first check the longest monotone sequence of P (similarly defined as before). In case this length is larger than $n^{7/13}$, we just color every second point of it with the same new color and proceed recursively on the remaining points. Since we can use $O(n^{6/13})$ colors and this step removes $\Omega(n^{7/13})$ points, per color, we remain within the desired bound (this can easily be formalized by solving the recurrence).

From now on, we assume that there is no monotone sequence of length larger than $n^{7/13}$ in P. Now we make a grid G_r with $r = n^{5/13}$, so that each row and column contains at most $B = n^{8/13}$ points.

Now we will define some "border" point sets. For a given set of points $Q \subseteq P$, let $D_i(Q)$ ($i = 1, 2, 3, 4$) consist of those points $q \in Q$ for which there is no other point of Q in the i-th quadrant relative to q. Also, let $D(Q) = \bigcup_{i=1}^{4} D_i(Q)$.

Let the points of P be partitioned by the grid cells into $R_i \cap C_j$ where R_i is the i-th row and C_j is the j-th column of G_r. Let $D = \bigcup_{i,j} D(R_i \cap C_j)$.

Let us now bound the size of D. We can separately look at each diagonal strip of G_r and check the border points $D_i(R_i \cap C_j)$ for the cells in that strip for the corresponding direction. By construction, we can form a monotone sequence by chaining these together along the diagonal, so the length of this sequence is at most $n^{7/13}$ by our assumption. Since we have $O(r) = O(n^{5/13})$ diagonals and 4 directions, we get $|D| = O(n^{12/13})$. Now we color D conflict-free using Theorem 3.1, this gives $O(n^{6/13})$ colors so far. Now we take $P' = P \setminus D$ and just note that it still holds for these points that in every row and column there are at most $B = n^{8/13}$ of them, so we can apply Lemma 3.2 to get a QCF coloring of P' with respect to G_r using $\tilde{O}(B^{3/4}) = \tilde{O}(n^{6/13})$ colors.

Of course we use a disjoint set of colors for D and P'. Now we check that this is a conflict-free coloring of P. Let T be some non-empty axis-parallel rectangle with at least 2 points.
Case 1: T spans at least 2 rows and 2 columns of G_r. Then T contains a full corner of some grid cell (i, j), so it must contain a point from $D_i(R_i \cap C_j) \subseteq D$ for some quadrant i. The conflict-free coloring of D will show a point with a unique color, and since D and P' use disjoint colors this point is unique in $T \cap P$ as well.
Case 2: T lies entirely within one row or column of G_r. Again, if $T \cap D \neq \emptyset$ we are done by the previous argument. Otherwise, $T \cap P = T \cap P'$, then the QCF coloring of P' gives us a unique color in $T \cap P'$ by definition.

This finishes the proof.

□

3.4 Further improvements

One observation is that in one step of the proof of Theorem 3.3 we just used the $O(\sqrt{n})$ bound from Theorem 3.1 as a black box to color D. Instead of that we can just plug in Theorem 3.3 itself recursively as the black box, which gives a better bound. When we iterate this idea of substituting the improved algorithm as the black box, we get an increasingly better sequence of algorithms. In [8] it is shown that this sequence converges to an $O(n^{0.382})$ algorithm.

Chan [3] further improved this to $O(n^{0.368})$ by using some more refined hierarchical decomposition instead of the grid which is not discussed here. This is the current best known upper bound.

3.5 Some other ideas

Earlier we observed that we can decompose our graph into 2 *Hasse diagrams*, as: $D(P) = H_1 \cup H_2$. This suggests a different strategy for the upper bound. If H_1 and H_2 are properly colored with c_1 and c_2 colors respectively, then we can create $c_1 \cdot c_2$ many colors by taking the Cartesian product of the color sets and then we assign to each point the pair of colors it receives in H_1 and H_2. Clearly $D(P)$ will be properly colored this way. Although Suk and Tomon [9] show that there exist Hasse diagrams with large chromatic number, more precisely, there is a Hasse diagram H for which $\chi(H) = \Omega(n^{1/4})$, it is not clear whether such a diagram can be realized as H_1 or H_2 for some point set P. So it is open whether $\chi(H_1) = O(\operatorname{polylog} n)$, and in case that is true, we immediately get $\chi(D(P)) = O(\operatorname{polylog} n)$ by the product coloring argument.

On another note, Ackerman and Pinchasi [10] showed that $O(\log n)$ colors suffice if we only require every axis-parallel rectangle containing at least *three* points of P to be non-monochromatic. Of course this is a slightly weaaker condition, but their method also suggests some ideas for the main problem.

4 Lower Bounds

4.1 The probabilistic construction

In general, it is hard to give explicit constructions of point sets with large chromatic number. Surprisingly, the best known result here comes from showing a lower bound for a random point set.

The lower bound in Theorem 1.1 comes from the fact that there exist point sets with small independence number, which forces a large chromatic number via (1). The key result is the following.

Theorem 4.1 (Chen-Pach-Szegedy-Tardos [1], Jin-Kwan-Lichev [4]). *There exist n-point sets P in general position with*

$$\alpha(D(P)) = O\left(\frac{n \log\log n}{\log n}\right).$$

The original result of Chen et al. [1] gave $\alpha(D(P)) = O(n(\log\log n)^2/\log n)$, which gives $\chi \geq \Omega(\log n/(\log\log n)^2)$. Jin, Kwan and Lichev [4] recently improved this to Theorem 4.1 with an additional observation, which is currently the best known bound in Theorem 1.1.

We sketch the main ideas of the proof here without the calculations. As it is mentioned earlier, we choose a random set of n points in $[0,1]^2$ and show that with high probability $\alpha(D(P))$ is small. Since $D(P)$ depends only on the relative order of coordinates, a random set of n points in $[0,1]^2$ is equivalent to choosing a random permutation π of $\{1,\ldots,n\}$, for the y-coordinates. We aim to show that with high probability every large subset $I \subseteq P$ contains an edge of $D(P)$.

Instead of choosing all y-coordinates at once, we generate them digit by digit in base L (to be chosen later) and reveal these digits one at a time for all points. We denote the y-coordinates as $y_i = (0.d_i^{(1)}d_i^{(2)}\ldots)_L$ where the digits are chosen independently and uniformly from $\{0,\ldots,L-1\}$. At step t we reveal digit $d_i^{(t)}$ for all i, and we use the following notation: $y_i^{(t)} = (0.d_i^{(1)}\ldots d_i^{(t-1)})_L$. We will do $O(\log_L n)$ steps in total, this will be enough to differentiate the coordinates with high probability (which can be formalized).

We say that the edge $\{p_i, p_j\}$ (with $i < j$) is forced at step t if:

1. $y_i^{(t)} = y_j^{(t)}$ (so p_i and p_j are at the same level),
2. $d_i^{(t)} = d_j^{(t)}$ (they receive the same new digit), and
3. $d_k^{(t)} \neq d_i^{(t)}$ for all k with $i < k < j$ satisfying $y_k^{(t)} = y_i^{(t)}$ (so no point between them in the same level gets the same digit).

Clearly, if this happens, p_i and p_j will have an edge between them in $D(P)$ regardless of the random digits to be revealed later. Not all edges of $D(P)$ are necessarily forced at some point, but in the paper it is enough to consider only forced edges to get a lower bound.

The next idea is to calculate the probabilities that our subset I survives each of the $O(\log_L n)$ steps without creating a forced edge. With careful calculation it can be shown that with $L = \lfloor \log n/(100\log^2\log n)\rfloor$ if the size of I is at least $3n/L$, then we will have a forced edge with high probability at some point. This gives $\alpha(D(P)) = O(n\log^2\log n/\log n)$, which then gives $\chi \geq \Omega(\log n/(\log\log n)^2)$. We can conclude from here that there must be such an exact construction that achieves this bound (without the randomness).

The result of Jin, Kwan and Lichev [4] uses the exact same proof idea with an observation that for points with indices i and j that are close, the bound for them to have a forced edge at some step can be strengthened. The calculation is not included here, but this can improve with a $\log\log n$ factor mentioned earlier.

This is a nice and surprising result, but in the Chen paper [1] it is shown that we cannot get a much better bound for this uniformly random construction: a random n-point set satisfies $\alpha(D(P)) = \Omega(n/\log n)$, so up to a $(\log\log n)$ factor the random construction is tight.

4.2 Suk-Tomon construction

Theorem 4.2 (Suk-Tomon [9]). *For every n, there exists a Hasse diagram on n vertices with $\alpha = O(n^{3/4})$ and $\chi = \Omega(n^{1/4})$.*

The idea of this construction is that we take lines and points on the plane and then make a graph where point-line incidences correspond to vertices. We have an edge from vertex (p_1, l_1) to (p_2, l_2) if p_1 also lies on l_2 and appears before p_2 along l_2 with no other incidence pair in between. Clearly, this gives us a covering relation, so this results in a Hasse-diagram.

As mentioned earlier, due to $D(P) = H_1 \cup H_2$, it is a closely related question to analyse the chromatic number of Hasse-diagrams. However we need to find a construction that is realized in 2d as H_1 to get an upper bound for our main problem.

As a side note, Damásdi found a construction of triangle-free graphs on n vertices with $\alpha = O(n^{2/3})$ using finite projective planes, which was recently discovered by Kocbek [11] as well. This is a closely related problem, but it does not involve Hasse diagrams.

5 Computational experiments

For verifying some construction ideas, we wrote a computer program in python to generate families of point sets with their $D(P)$ graph and then check the independence number. It is relatively cheap to check constructions up to $n \leq 100$ points, which can give us suggestive results. The conclusion is that it is possible to beat the uniform random construction, but not by a significant amount, and it is hard to reason about the chromatic numbers for general n. These small brute force trials suggest that polylogarithmic number of colors should always be enough. See appendix for some of the concrete ideas that were tried.

6 Final Remarks

The main open question remains whether any of the bounds in Theorem 1.1 can be further improved. It is widely believed that there is a polylogarithmic upper bound. However, all known upper bound techniques hit a polynomial barrier as of today. One of the most promising direction for making an improvement from both directions is to better understand 2-dimensional Hasse diagrams. The Suk-Tomon construction shows that we must use the 2-dimensional property for proving the polylogarithmic upper bound conjecture.

Appendix

A Python tool for calculating max independent sets

Figure 1 shows an example of a random point set with $n = 36$ points together with a maximum independent set of $D(P)$ highlighted in orange. This tool was made with Python to experiment with different families of point sets.

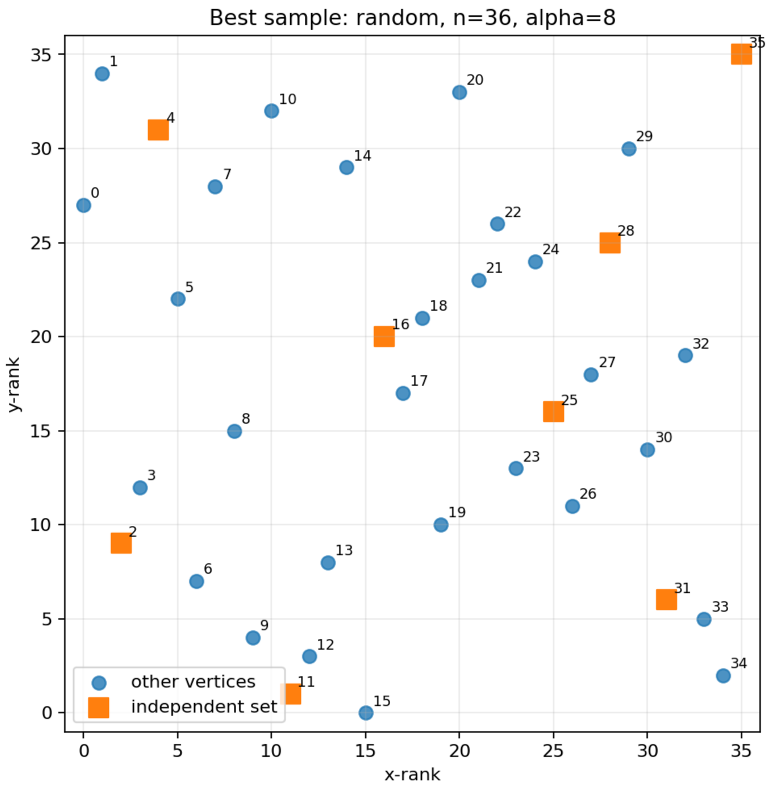

Figure 1: A random point set with $n = 36$, orange squares form a maximum independent set of size $\alpha = 8$.

B Comparison of construction families

We include a few of the families here that we generated and for which we calculated $\alpha(D(P))$ (the lower the better) across different values of n:

random: (uniformly random permutation, the one for which the best known lower bound is proven);

grid scramble: ($\sqrt{n} \times \sqrt{n}$ grid with rows and columns independently shuffled uniformly randomly);

greedy: (builds the permutation one element at a time minimizing the next current value of α, it chooses randomly for tie-breaking);

irrational rotation: (y-ranks given by sorted fractional parts of $i \cdot \phi$, $\phi = (\sqrt{5} - 1)/2$). (in general these kind of constructions turn out to give us grid-like structures for which it is easy to have large independent sets unfortunately)

The results are shown in Figure 2. It seems that greedy strategies can beat random constructions, but not by a significant amount, and it remains hard to reason about bounds for them.

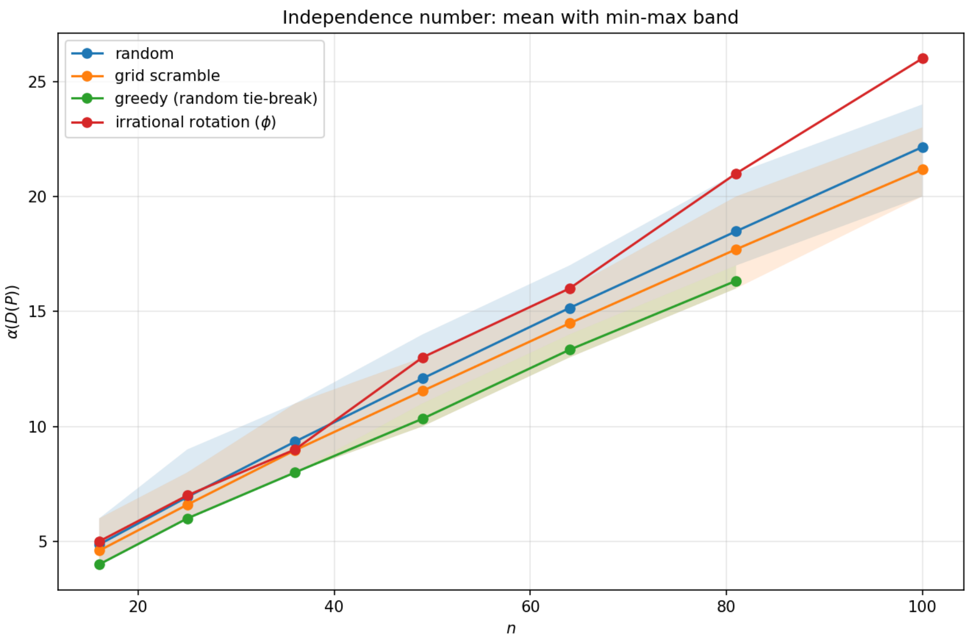

Figure 2: Mean $\alpha(D(P))$ with min-max band for four permutation families.

References

[1] X. Chen, J. Pach, M. Szegedy, G. Tardos. Delaunay graphs of point sets in the plane with respect to axis-parallel rectangles. *Random Structures & Algorithms*, 34(1):11–23, 2009.

[2] G. Even, Z. Lotker, D. Ron, S. Smorodinsky. Conflict-free colorings of simple geometric regions with applications to frequency assignment in cellular networks. *SIAM Journal on Computing*, 33(1):94–136, 2003.

[3] T. M. Chan. Conflict-free coloring of points with respect to rectangles and approximation algorithms for discrete independent set. *Proc. 28th Annual Symposium on Computational Geometry (SoCG)*, pp. 293–302, 2012.

[4] Z. Jin, M. Kwan, L. Lichev. Colouring random Hasse diagrams and box-Delaunay graphs. arXiv:2501.12373, 2025.

[5] G. Damásdi. Coloring questions on axis-parallel rectangles and arithmetic progressions. arXiv:2602.18235, 2026.

[6] S. Har-Peled, S. Smorodinsky. Conflict-free coloring of points and simple regions in the plane. *Discrete & Computational Geometry*, 34(1):47–70, 2005.

[7] K. Elbassioni, N. Mustafa. Conflict-free colorings of rectangle ranges. *Proc. 23rd International Symposium on Theoretical Aspects of Computer Science (STACS)*, pp. 254–263, 2006.

[8] D. Ajwani, K. Elbassioni, S. Govindarajan, S. Ray. Conflict-free coloring for rectangle ranges using $O(n^{.382})$ colors. *Discrete & Computational Geometry*, 48(1):39–52, 2012.

[9] A. Suk, I. Tomon. Hasse diagrams with large chromatic number. *Bulletin of the London Mathematical Society*, 53(3):747–758, 2021.

[10] E. Ackerman, R. Pinchasi. On coloring points with respect to rectangles. *Journal of Combinatorial Theory, Series A*, 120(4):811–815, 2013.

[11] M. Kocbek. Explicit geometric construction of Ramsey graphs. arXiv:2507.09235, 2025.