Chromatic Number of the Delaunay Graph with Respect to Axis-Parallel Rectangles

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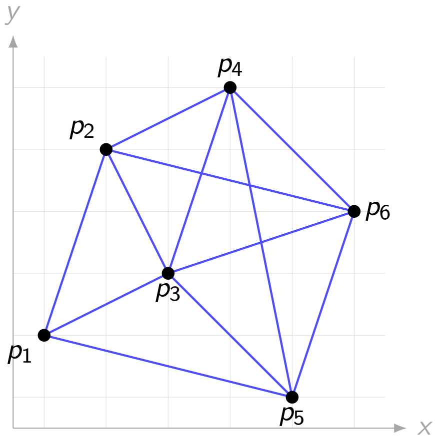
2026

Definition of $D(P)$

- P: finite point set in the plane in general position
- $D(P)$: graph on vertex set P, two points $p, q \in P$ are connected if and only if the smallest axis-parallel rectangle with opposite corners p and q contains no other point of P.



Chromatic number of $D(P)$

- What can we say about $\chi(D(P))$?
  - motivation: wireless frequency assignment, Smorodinsky et al., 2003

Theorem (Chen-Pach-Szegedy-Tardos 2009, Chan 2012, Jin-Kwan-Lichev 2025)

For every set P of n points in general position,

$$\chi(D(P)) \leq O(n^{0.368})$$

There exists a set P of n points for which:

$$\Omega\left(\frac{\log n}{\log \log n}\right) \leq \chi(D(P))$$

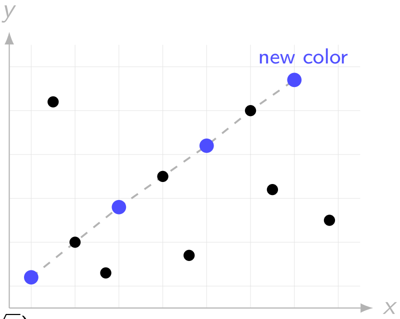
- JKL proves the lower bound (improving CPST)
- Chan proves the upper bound
- Gap is large!!!

Upper bounds

Claim:

$$\chi(D(P)) = O(\sqrt{n})$$

- We look at the y coordinates in the order of x coordinates
- Erdős-Szekeres gives a monotone subsequence of length at least $\sqrt{n}$.
- Color every second point the same and recurse on the rest



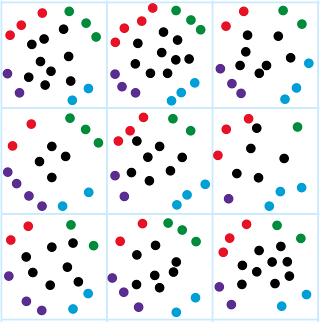
$$f(n) \leq 1 + f\left(n - \left\lfloor \frac{\sqrt{n}}{2} \right\rfloor\right), \qquad f(n) = O(\sqrt{n}).$$

Stronger upper bounds

- **Conflict-free (CF) coloring:** every nonempty axis-parallel rectangle contains a point whose color is unique inside the rectangle (stronger than proper coloring)

  Now we put the point set P inside an $r \times r$ grid:

- **Quasi-conflict-free (QCF) coloring:** every rectangle contained in a single row or column of the grid is conflict-free.
- **Border** points from 4 directions:



Stronger upper bounds

- Conflict-free coloring with respect to axis-parallel rectangles:

$$O(\sqrt{n}) \qquad \text{(Har-Peled-Smorodinsky, 2005)}$$

- QCF coloring with respect to a grid, assuming every row and column has at most B points:

$$\widetilde{O}(B^{3/4}) \qquad \text{(Elbassioni-Mustafa, 2006)}$$

- From these, we can prove:

$$\widetilde{O}(n^{6/13}) \qquad \text{(Ajwani-Elbassioni-Govindarajan-Ray, 2012)}$$

$\widetilde{O}(n^{6/13})$ proof sketch

- Monotone subsequence of size at least $n^{7/13}$: color every second point and recurse...
- Otherwise, $r = n^{5/13}$ and put P to an $r \times r$ grid, each row and column contains max. $B = n^{8/13}$ points.
- There are $O(n^{5/13})$ diagonals in the grid, each can have just $n^{7/13}$ border points for each direction => $O(n^{12/13})$ border points.
- Color D conflict-free using $O(\sqrt{n^{12/13}}) = O(n^{6/13})$
- Color $P \setminus D$ using QCF coloring:

$$\widetilde{O}(B^{3/4}) = \widetilde{O}\left((n^{8/13})^{3/4}\right) = \widetilde{O}(n^{6/13})$$

- ...easy to check from the picture that this works

Further upper bound ideas

- Chan improves this to

$$O(n^{0.368}) \qquad \text{(Chan, 2012)}$$

  (similar ideas, but long proof)

- Another idea: split $D(P)$ into two Hasse diagrams

$$D(P) = H_1 \cup H_2.$$

- Informally:
  - H_1: up-right going edges,
  - H_2: up-left going edges.
- It would be enough to find a bound on just H_1 and then use the product coloring

Lower bounds

- ...just choose the y-coordinates as a uniform random permutation...and that's the best we know (surprising!)
- We can just prove that the independence number of $D(P)$ is small, as we have:
$$\chi(D(P)) \geq \frac{n}{\alpha(D(P))}.$$

Jin-Kwan-Lichev 2025, (Chen-Pach-Szegedy-Tardos 2009)

For this uniform random P, with high probability:

$$\alpha(D(P)) = O\left(\frac{n \log\log n}{\log n}\right).$$

There exist point sets P with:

$$\chi(D(P)) \geq \frac{n}{\alpha(D(P))} = \Omega\left(\frac{\log n}{\log\log n}\right).$$

Random construction proof idea

- The point set is just a permutation...
- Reveal y coordinates digit by digit (in base L chosen suitably)

	p_1	p_2	p_3	p_4
start	0	0	0	0
step 1	0.2	0.2	0.3	0.2
step 2	0.25	0.23	0.31	0.25
step 3	0.254	0.238	0.312	0.251

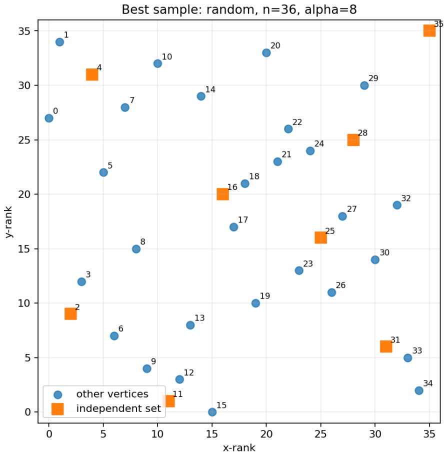
Random construction proof idea

- After step 2, p_1 and p_4 are still on the same level, but no point in between is on that level, so $p_1 p_4$ becomes a **forced edge**
- Proof only considers forced edges

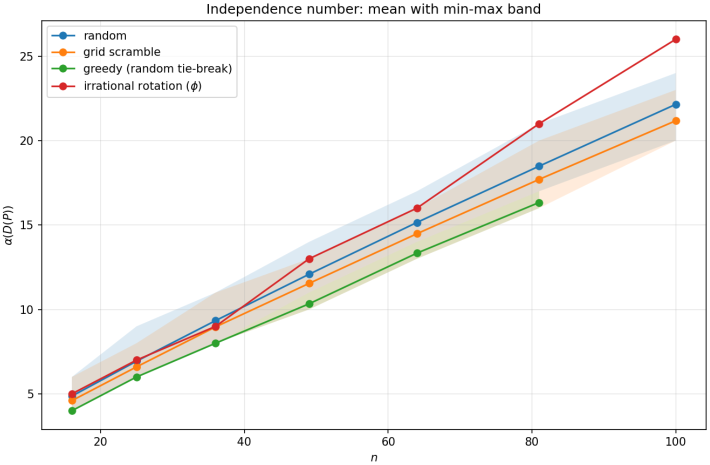
	p_1	p_2	p_3	p_4	
start	0	0	0	0	
step 1	0.2	0.2	0.3	0.2	
step 2	0.25	0.23	0.31	0.25	forced edge
step 3	0.254	0.238	0.312	0.251	

Computational experiments

- Generating construction families and analysing these with python scripts ($n \leq 100$ is feasible), easier to think about independent sets



Comparison of constructions



- Hard to beat random at all, even for small fixed n

Relevant papers

Upper bounds

- Har-Peled-Smorodinsky 2005
- Elbassioni-Mustafa 2006
- Ajwani-Elbassioni-Govindarajan-Ray 2012
- Chan 2012

Lower bounds

- Chen-Pach-Szegedy-Tardos 2009
- Jin-Kwan-Lichev 2025

Other related papers

- Even-Lotker-Ron-Smorodinsky 2003
- Ackerman-Pinchasi 2013
- Suk-Tomon 2021

AI usage

ChatGPT: review grammar/format, generate some TikZ in this presentation
Claude: help with generating some python scripts for lower bound constructions

Thank you!

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