Graph Matroid Families

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Introduction

Graph matroid families, or matroidal families assign a matroid to every graph in an isomorphism-preserving and restriction-compatible manner. Well-known examples of such families are the graphic matroid or even-cycle matroid of a graph. The family of generic d-dimensional rigidity matroids gives a useful perspective for matroid families. Last semester we investigated vertex matroid families, this semester we focused on the structural behavior of graph matroid families. In this report, we first introduce graph matroid families and standard notation. In the preliminaries, we state and prove useful claims from [1]. Then, we introduce this semester's particular question, that is, whether a special vertex splitting operation can maintain $\mathcal{M}$-independence, and show our partial results.

Definition 1. *A* ***graph matroid family*** $\mathcal{M}$ *is a family of matroids, defined on the edge set of each (finite, simple) graph in a way that is*

- *well defined: every graph isomorphism* $\varphi : V(G) \to V(H)$ *induces an isomorphism between* $\mathcal{M}(G)$ *and* $\mathcal{M}(H)$*.*
- *compatible: for all subgraphs* H *of* G*,* $\mathcal{M}(H)$ *is a restriction of* $\mathcal{M}(G)$*.*

Let $\mathcal{M}$ be a graph matroid family. We call a graph G $\mathcal{M}$-independent if its respective matroid $\mathcal{M}(G)$ is the free matroid on $E(G)$ and $\mathcal{M}$-dependent otherwise. We say that G is an $\mathcal{M}$-circuit if $\mathcal{M}(G)$ is dependent but for all edges $e \in E(G)$, $\mathcal{M}(G - e)$ is $\mathcal{M}$-independent. We define the rank function r of $\mathcal{M}$ by letting $r(G)$ be the size of maximal independednt set in G, for each graph G.

We call an edge $e \in E(G)$ a bridge if there exists no subgraph G' of G such that $e \in G'$ and $\mathcal{M}(G')$ is an $\mathcal{M}$-circuit. Equivalently, a bridge is contained in every basis of the matroid of the graph.

We say that a graph matroid family $\mathcal{M}$ is trivial if every graph is $\mathcal{M}$-independent (which means that for all graph G $\mathcal{M}(G)$ is the free matroid), otherwise $\mathcal{M}$ is nontrivial. For a nontrivial graph matroid family we define its dimensionality $d_{\mathcal{M}}$ and threshold $t_{\mathcal{M}}$ the following way:

$$d_{\mathcal{M}} = \min\{d : \text{ there exists an } \mathcal{M}\text{-circuit with minimum degree } d+1\}$$

$$t_{\mathcal{M}} = \min\{|V(C)| - 1 : \ C \text{ is an } \mathcal{M}\text{-circuit with minimum degree } d+1\}$$

Although these definitions may initially appear arbitrary, in Lemma 1 it is shown that the dimensionality determines the growth rate of the rank function once the number of vertices exceeds the threshold.

Preliminaries

The rank function of a graph matroid family exhibits a highly regular structure; specifically, once the number of vertices surpasses the threshold, the rank of complete graphs scales linearly with respect to the number of vertices.

Lemma 1. *(Linearity of rank) [1] Let $\mathcal{M}$ be a nontrivial graph matroid family with rank function r, dimensionality d and threshold t. We have $r(K_n) = d(n-t) + r(K_t)$.*

Proof. Fix $n \geq t$. It suffices to show that $r(K_{n+1}) = r(K_n) + d$. Consider K_n and add a new vertex v and connect it with d other vertices. Call this graph G. Since the degree of v is exactly d it cannot be a part of any $\mathcal{M}$-circuit, thus each edge on it is a bridge. From this follows that $r(G) = r(K_n) + d$.

Fix an $\mathcal{M}$-circuit C on t vertices and with minimum degree $d+1$. We have $r(K_{n+1}) = r(G)$ since by adding any new edges to G we cannot increase the rank, since it is contained in a copy of C. It follows that $r(K_{n+1}) = = r(K_n) + d$. □

We say that a graph matroid family $\mathcal{M}$ is unbounded if the series $r(K_n)$ with $n \to \infty$ is unbounded, otherwise we call it bounded. From the previous lemma we can see that $\mathcal{M}$ is bounded if and only if $d_{\mathcal{M}} = 0$.

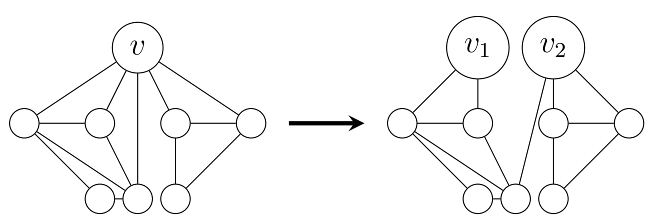
Lemma 2. *[1] A graph matroid family $\mathcal{M}$ is unbounded if and only if every forest is $\mathcal{M}$-independent.*

Proof. If $\mathcal{M}$ is unbounded, then by definition every $\mathcal{M}$-circuit has minimum degree at least 2, and thus every forest is $\mathcal{M}$-independent. Conversely, if $\mathcal{M}$ is bounded, then $\exists N$ such that $\forall n > N$ $r(K_n) = r(K_N)$. Consider a forest consisting of $r(K_N) + 1$ edges. It must be $\mathcal{M}$-dependent. □

Corollary 1 (Vertex addition)**.** *[1] If $\mathcal{M}$ is a nontrivial graph matroid family with dimensionality d, then the addition of vertices of degree at most d preserves $\mathcal{M}$-independence. More generally, if G is a graph and $v \in V(G)$ is a vertex with $d_G(v) \leq d$, then every edge incident to v is a bridge in $\mathcal{M}(G)$.*

This semester's question

We can see in Corollary 1 that by adding a new vertex with at most d edges connecting it to the original graph preserves $\mathcal{M}$-independence and thus we can make new $\mathcal{M}$-independent graphs from previously $\mathcal{M}$-independent graphs. A natural question is whether we can define other operations that preserve $\mathcal{M}$-independence. The idea of the operation of splitting a vertex into two vertices arises naturally and was asked by Garamvölgyi, the author of [1] whether it always preserves independence. This operation takes a vertex v and replaces it with two other vertices v_1 and v_2. The edges formerly connected to v are then reattached to either v_1 or v_2.



This natural operation preserves $\mathcal{M}$-independence in all known graph matroid families. Although we know that there are uncountably many graph matroid families [3], almost all of them have a naturally nice structure.

We investigated the question whether we can obtain from an $\mathcal{M}$-independent graph G an $\mathcal{M}$-circuit by splitting the vertices. This perspective proves advantageous, as the circuit elimination axiom (and its strong version) shows to be an effective tool for investigating graph matroid families.

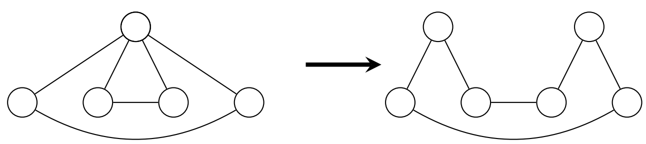
Question 1. *Let G be an $\mathcal{M}$-independent graph, $v \in V(G)$. Can we split the vertex into two new vertices v_1 and v_2 and split its original edges among the new vertices such that the new graph is $\mathcal{M}$-dependent?*

We managed to prove that the question is not true in general.

Claim 1. *There exist bounded graph matroid families for which this splitting of a vertex creates an $\mathcal{M}$-dependent graph from an $\mathcal{M}$-independent graph.*

A simple observation is that we may assume that after the splitting we obtain an $\mathcal{M}$-circuit. Notice that the subdivided vertices must be the part of an $\mathcal{M}$-circuit (otherwise the original graph is also $\mathcal{M}$-dependent). By removing edges from the graph we can obtain an $\mathcal{M}$-circuit which contatins the two subdivided vertices. In the original graph, we can remove the same edges, and since it was originally $\mathcal{M}$-independent, it remains so.

Example 1. *A minimal example is the graph matroid family where each graph with 6 or less edges are independent except for the C_6 (cycle of length 6) graph, and every graph on at least 7 edges is $\mathcal{M}$-dependent.*



It is easy to see that this is indeed a graph matroid family. This matroid family is obviously bounded. This boundedness helps us create this example, which gives us more freedom to shape the matroid family to our liking and obtain a paving matroid-like structure.

Remark 1. *The other direction is trivially true, that is, if we have an $\mathcal{M}$-dependent graph and we subdivide a vertex into two other vertices, we may obtain a $\mathcal{M}$-independent graph. Consider an $\mathcal{M}$-circuit with exactly one vertex with degree $d_{\mathcal{M}}+1$ and subdivide it. By the definition of $d_{\mathcal{M}}$, the new graph cannot be $\mathcal{M}$-dependent. Another easy example is just considering the graphic matroids of graphs, and subdividing a vertex of a cycle C_k, thus obtaining a path.*

The unbounded case is much more complicated. While considering the unbounded case and assuming we have some $\mathcal{M}$-circuit, we have to consider what other $\mathcal{M}$-dependent sets and $\mathcal{M}$-circuits the graph "generates". It may happen that it generates its unsplit version. This is the case for smaller examples for the bounded case and if for all unbounded graph matroid families this is true, then we have a negative answer fo the question. Further hardness comes from the fact that unlike in the bounded counterexample, in the general setting the graph may have several other vertices not connected to the split vertices. For the currently known graph matroid families, the vertex splitting operation seems to maintain $\mathcal{M}$-independence, but most of these families have motivation from other aspects of mathematics which may grant them structure which the general matroid families do not have.

We started understanding the unbounded case with the smallest possible case, where we split a 4-degree vertex into two 2-degree vertices. Even this case proves to be quite complicated. We investigated, whether we can achieve some results by gluing $\mathcal{M}$-circuits together and use the circuit elimination on them. This direction did not provide notable results yet.

References

[1] Dániel Garamvölgyi. Rigidity and reconstruction in matroids of highly connected graphs, 2024.

[2] Laurence R. Matthews. Matroids from directed graphs. *Discrete Mathematics*, 24(1):47–61, 1978.

[3] Rüdiger Schmidt. On the existence of uncountably many matroidal families. *Discrete Mathematics*, 27(1):93–97, 1979.