Multitype branching processes

for modeling complex contagion on social networks

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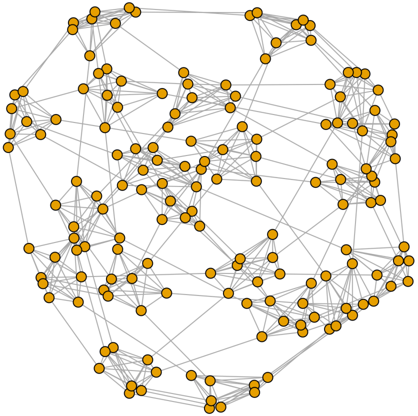
Motivation and Background

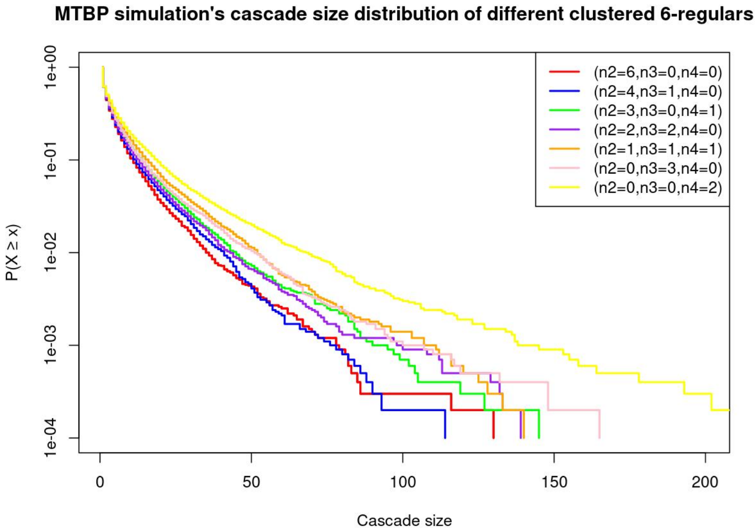
- Centola's experiment on social contagion
- Complex contagion:
  $p_1 := \mathbb{P}(\text{A node getting infected from the first active node contact})$,
  $q_1 := \mathbb{P}(\text{A node not getting infected from the first contact}) = 1 - p_1$,
  $q_k := \mathbb{P}(\text{A node not getting infected from the } k\text{th contact}) = q_1(1-\alpha)^{k-1}$,
  $p_k := \mathbb{P}(\text{A node gets infected from the } k\text{th contact}) = 1 - q_k$
- Newman-Miller graphs
- The study of Keating, Gleeson and O'Sullivan: Multitype branching process (MTBP) using motifs to simulate

Previous Semester's Work: Newman–Miller Graphs

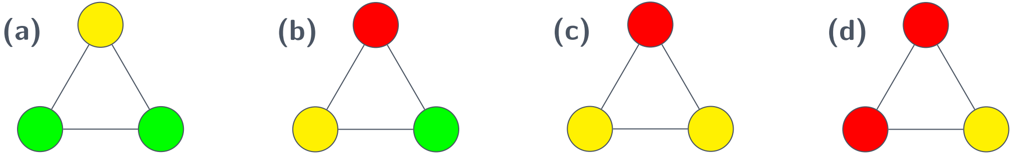
Example of a graph in which every vertex belongs to a K_2 and a K_8



This Semester's Work: MTBP Simulations



MTBP on a K_3



Motif	Created offspring	Probability
(a) $\to \emptyset$	0	$(1-p_1)^2$
(a) $\to$ (b)	$(n-1)(a)+(b)$	$2p_1(1-p_1)$
(a) $\to$ (c)	$2(n-1)(a)+(c)$	p_1^2
(b) $\to \emptyset$	0	$(1-p_1)(1-\alpha)$
(b) $\to$ (d)	$(n-1)(a)+(d)$	$\alpha+p_1(1-\alpha)$

Mean Matrix for a Network Having K_3's

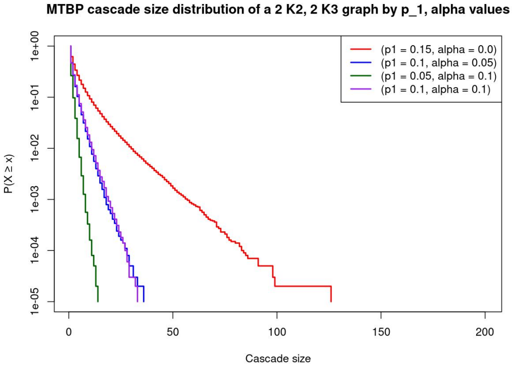
$$\mathrm{M}^3 = \begin{pmatrix} 2p_1(n-1) & (n-1)(\alpha + p_1(1-\alpha)) & 0 & 0 \\ 2p_1(1-p_1) & 0 & 0 & 0 \\ p_1^2 & 0 & 0 & 0 \\ 0 & \alpha + p_1(1-\alpha) & 0 & 0 \end{pmatrix}$$

Mean Matrix for a Network Having K_2's, K_3's and K_4's

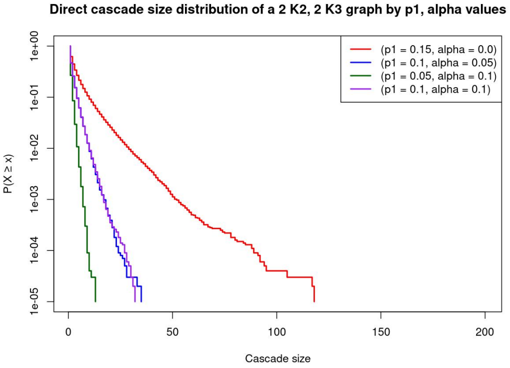
$$x := \alpha + p_1(1-\alpha), \quad y := 1-(1-p_1)^2(1-\alpha)^3, \quad z := 1-(1-p_1)(1-\alpha)^2.$$

$$M^{2,3,4} = \begin{pmatrix}
3(n_4-1)p_1 & 2(n_4-1)x & (n_4-1)y & 0 & (n_4-1)z & 0 & 0 & 2n_4p_1 & n_4x & 0 & 0 & n_4p_1 & 0 \\
3p_1(1-p_1)^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3p_1^2(1-p_1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_1^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2(1-p_1)(1-\alpha)x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & y & 0 & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3n_3p_1 & 2n_3x & n_3y & 0 & n_3z & 0 & 0 & 2(n_3-1)p_1 & (n_3-1)x & 0 & 0 & n_3p_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2p_1(1-p_1) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p_1^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 \\
3n_2p_1 & 2n_2x & n_2y & 0 & n_2z & 0 & 0 & 2n_2p_1 & n_2x & 0 & 0 & (n_2-1)p_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_1 & 0
\end{pmatrix}$$

MTBP vs Direct Simulations with Different p_1 and α Values

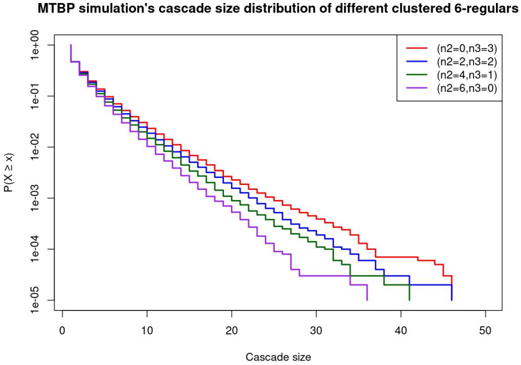


Approximating MTBP simulations

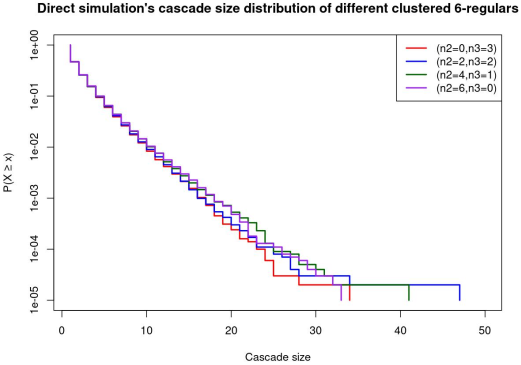


Direct simulations on the generated graph

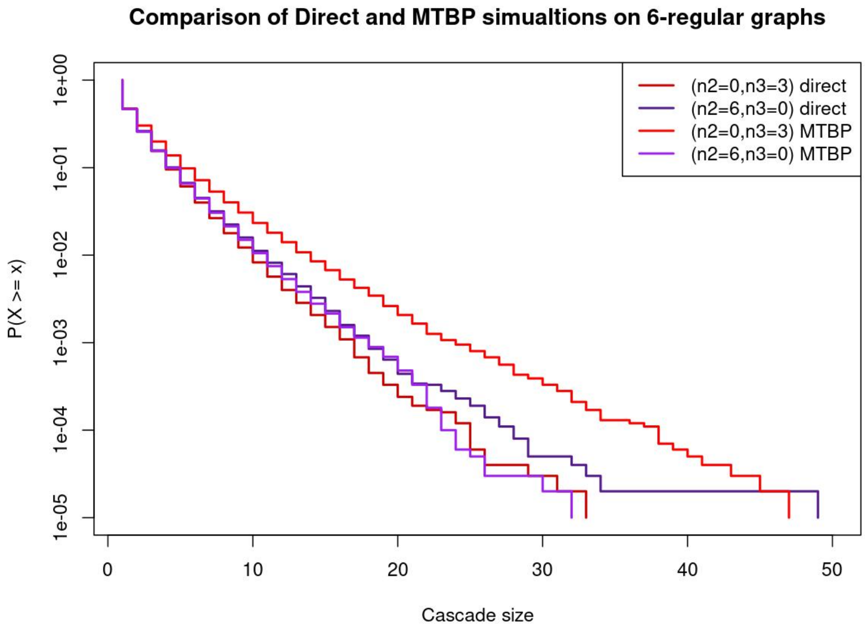
MTBP vs Direct Simulations on 6-Regular Networks



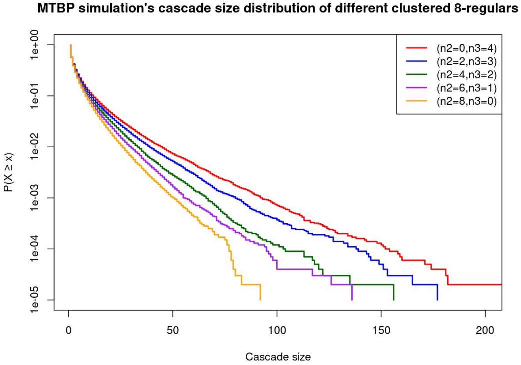
MTBP simulations



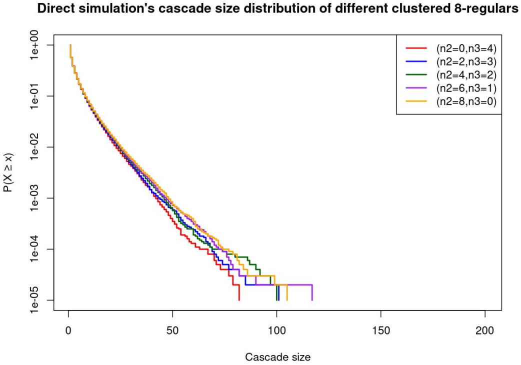
Direct simulations


Comparison of Direct and MTBP simualtions on 6-regular graphs
(n2=0,n3=3) direct
(n2=6,n3=0) direct
(n2=0,n3=3) MTBP
(n2=6,n3=0) MTBP
P(X >= x)
1e+00
1e-01
1e-02
1e-03
1e-04
1e-05
0
10
20
30
40
50
Cascade size

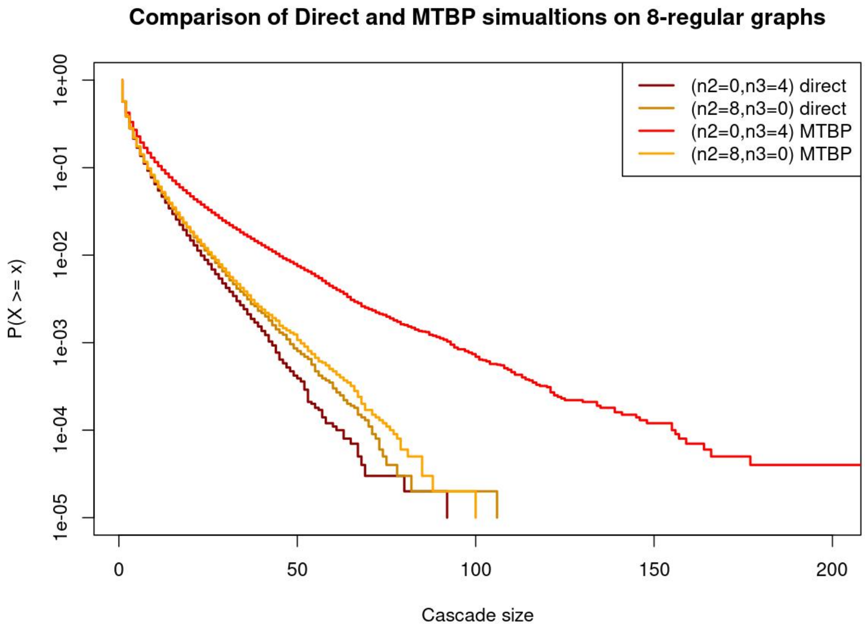
MTBP vs Direct Simulations on 8-Regular Networks



MTBP simulations



Direct simulations

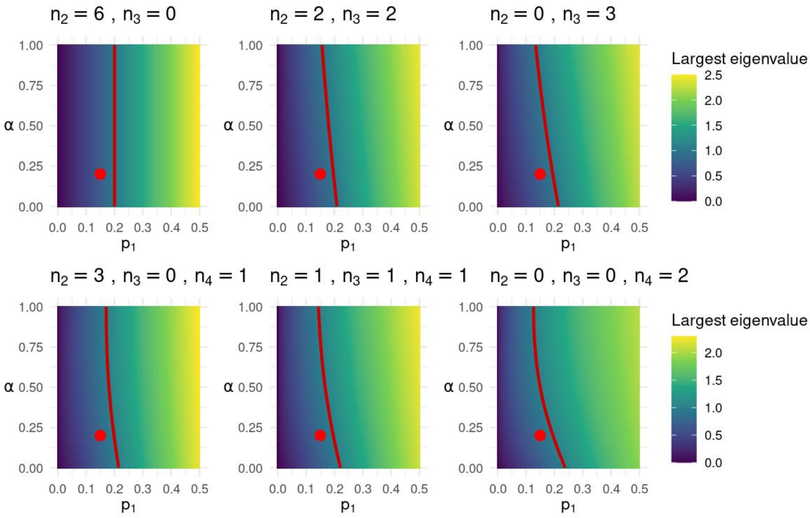


Comparison of Direct and MTBP simualtions on 8-regular graphs
(n2=0,n3=4) direct
(n2=8,n3=0) direct
(n2=0,n3=4) MTBP
(n2=8,n3=0) MTBP
P(X >= x)
1e+00
1e-01
1e-02
1e-03
1e-04
1e-05
0
50
100
150
200
Cascade size

Eigenvalues of Newman-Miller Graphs

- Given the mean matrix M[2,3,4]
- Can be rearranged into block form and reduced.
- The critical behavior of the contagion process is determined by the largest eigenvalue: λ_{max}
  - **subcritical** (it dies out with a probability of 1) if $\lambda_{max} < 1$
  - **supercritical** (meaning the cascade size tends to infinity with positive probability) if $\lambda_{max} > 1$
  - **critical** if $\lambda_{max} = 1$

The Largest Eigenvalue



Correlations of the clustering coefficient and the largest eigenvalue

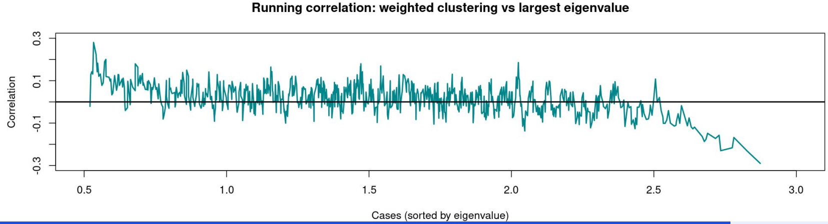
The clustering coefficient:

$$C = \frac{\text{number of closed triplets}}{\text{number of all triplets (open and closed)}} = \frac{2(n_3 + 3n_4)}{(3n_4 + 2n_3 + n_2)(3n_4 + 2n_3 + n_2 - 1)}$$

The weighted clustering coefficient:

$$C_w(n_2, n_3, n_4) = \frac{2\,(n_3 + 3n_4)}{3n_4 + 2n_3 + n_2 - 1}$$

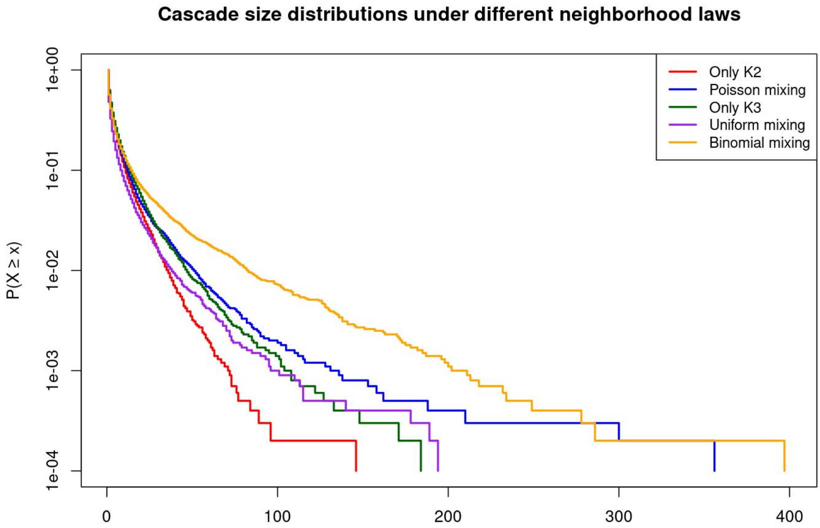
Correlations Between the Largest Eigenvalues and the Weighted Clustering Coefficients in Networks:



Future Work

- Analytical explanation of simulation results
- Theory from the recent article by Keating, Gleeson, and O'Sullivan
  - Generating functions
- Different types of edges
- Networks with heterogeneous local structures

Future Work: Different Local Structures



Thank you for your attention

References

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L. A. Keating, J. P. Gleeson, and D. J. P. O'Sullivan.
A generating-function approach to modelling complex contagion on clustered networks with multi-type branching processes.
Journal of Complex Networks, 11(6):cnad042, 2023.

Use of AI Tools

AI-based tools were used for

- improving wording and structure of the presentation and summary,
- LaTeX formatting and layout assistance,
- programming.