Applications of arborescence packing

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My goal in this short note is to show a part of the topic of *packing time-respecting arborescences*, and present matroid-based proofs for the main theorems, however, their proofs are not complete yet.

Given a directed graph $D = (V \cup s, A)$ with a root vertex s with zero indegree. We say that D is a pre-flow, if $|\varrho_D(v)| \geq |\delta_D(v)|$ for every $v \in V$. A packing of arborescences means a set of arc-disjoint arborescences. $\lambda_D(s, v)$ denotes the maximum number of arc-disjoint (s, v)-paths in D, where an (s, v)-path is a directed path from s to v. We say that $D' = (V \cup \{s, t\}, A)$ is almost Eulerian, if for every vertex $v \in V$ $|\varrho_{D'}(v)| = |\delta_{D'}(v)|$, and $|\varrho_{D'}(s)| = 0 = |\delta_{D'}(t)|$.

Also, given a time label function $\tau : A \to \mathbb{N}$, and we call the $N = (D, \tau)$ pair a temporal network. For an arc a, $\tau(a)$ is the time when the tail of a can transmit some information to the head of a. Temporal networks were introduced to model the exchange of information in a network or show the spread of a disease in a population. Let $\varrho^i_N(v) := \{a \in \varrho_D(v) : \tau(a) \leq i\}$ and $\delta^i_N(v) := \{a \in \delta_D(v) : \tau(a) \leq i\}$, then we call the temporal network N pre-flow if $|\varrho^i_N(v)| \geq |\delta^i_N(v)|$ for all $i \in \mathbb{N}$ and $v \in V$, and N is acyclic, if D is acyclic. (D, τ) is consistent if arcs of different τ-values cannot belong to the same strongly connected component of D. A directed path P of D, with arcs $a_1, ..., a_l$ in this order, is called time-respecting (τ-respecting) if $\tau(a_i) \leq \tau(a_{i+1})$ for $1 \leq i \leq l - 1$. An s-arborescence T is called time-respecting if for every vertex v of T, the unique (s, v)-path in T is time-respecting. The definition of $\lambda_N(s, v)$ is similar to the previous, but here the (s, v)-paths are time-respecting. If $D' = (V \cup \{s, t\}, A')$ is almost Eulerian, and $N' = (D', \tau')$ is a temporal network, then for every vertex $v \in V$, we call a bijection $\mu'_v : \delta_{D'}(v) \to \varrho_{D'}(v)$ τ'-respecting if for every $f \in \delta_{D'}(v)$ we have $\tau'(\mu'_v(f)) \leq \tau'(f)$.

For acyclic temporal networks there is an algorithm to get arc-disjoint time-respecting s-arborescences, if some conditions hold. The "*general*" case is harder, but there is a similar theorem to *Edmonds'* theorems.

Theorem 1. *[2] Let $N = ((V \cup s, A), \tau)$ be a consistent pre-flow temporal network. There exists a packing of $|\delta_D(s)|$ τ-respecting s-arborescences s.t. each vertex $v \in V$ belongs to $\lambda_N(s, v)$ of them.*

To prove this, we need the "*basic form*" of this theorem. The main idea is in [2], they create an acyclic graph by contracting the strongly connected components. Because of Theorem 2., there will be good arborescences in the strongly connected components. In the acyclic graph there are some special paths, and if we blow up the contracted components, then the arborescences can be added to the paths and then we get the arborescences of Theorem 1.

Theorem 2. *[2], [1] Let $D = (V \cup s, A)$ be a pre-flow directed graph, and $k := \max\{\lambda_D(s, v) : v \in V\}$. There exists a packing of k s-arborescences s.t. each vertex $v \in V$ belongs to at least $\lambda_D(s, v)$ of them.*

From now on we will prove this theorem by using matroid theoretical ideas. The original idea was to give a proof which uses matroid-rooted arborescences. Here the proof will be more similar to the non-matroid proof.

Let $\mathcal{M}$ be the following matroid. We take the graph $D = (V \cup s, A)$, and let $\mathcal{M}_s$ be a matroid on the root-edges $\overrightarrow{sv}$ with rank function r, all the other edges of D are the elements of a free matroid $\mathcal{M}_{free}$, and $\mathcal{M}$ is the direct sum of these two matroids. Let $\varrho^r(X) := \varrho_{free}(X) + r(\varrho_s(X))$, where $\varrho_{free}(X)$ is the number of free-edges going into X, and $\varrho_s(X)$ is the set of root-edges going into X. Let $\lambda_{\mathcal{M}}(s, v)$ be the maximum rank of the root-edges of the arc-disjoint sv paths. We want to split off edges, but in the previous form of the matroid it is complicated, because of the root-edges, so we split every edge $\overrightarrow{sv}$ into edges $\overrightarrow{sv'}$ and $\overrightarrow{v'v}$, let S be the set of these new vertices, and call the new graph D. The edges $\overrightarrow{sv'}$ "are" the elements of $\mathcal{M}_s$ and edges $\overrightarrow{v'v}$ are also the elements of the matroid $\mathcal{M}_{free}$. In this case, if $X \subseteq V$, then $\varrho^r(X) = \varrho_D(X)$, because every incoming edge is from the free matroid. Beacuse we only care about the original vertices and $\lambda_{\mathcal{M}}$ values, we can always assume that $X \cap S = \emptyset$. If we split off edges $\overrightarrow{u_1v}$ and $\overrightarrow{vu_2}$, where $u_1, u_2, v \notin S$, then $\varrho^r(X)$ ($v \notin X$) will decrease with one (free) edge if $u_1, u_2 \in X$, otherwise if $u_1 \in X$, $u_2 \notin X$ or $u_1, u_2 \notin X$, then the indegree will not change, and if only u_2 is in X, then the indegree will decrease with one because of $\overrightarrow{vu_2}$, but will increase with one because of the new $\overrightarrow{u_1u_2}$.

We want to prove the following extension of the theorem:

Extension: In the above defined matroid $\mathcal{M}$ we can pack arc-disjoint s-arborescences where every vertex will be covered by at least $\lambda_{\mathcal{M}}(s,v)$ arborescences.

Proof. We use induction on the number of vertices. If $|V|=1$, then it is trivial ($k=\lambda_{\mathcal{M}}(s,v)$ where $v \neq s$). So suppose $|V| \geq 2$, and there is no incoming edge into s. Let $v^* \in V$ be the vertex which has the smallest $\lambda_{\mathcal{M}}(s,v)$ value (if there are more, choose one arbitrarily). In the original proof ([1]), since D is pre-flow (with the new set S too) they could add new edges: for every $v \in V$ we add $\varrho_D(v)-\delta_D(v) \geq 0$ copies of the edge $\overrightarrow{vs}$, so the graph is Eulerian (if every vertex except s has the same in and outdegree, then $\varrho_D(s)=\delta_D(s)$), so they could apply another theorem and they could assume that there are $e=\overrightarrow{u_1v^*}$ and $f=\overrightarrow{v^*u_2}$ edges which can be split off such that in the new graph D' $\lambda_{D'}(s,v)=\lambda_D(s,v)$ for every $v \in V-v^*$, and $\lambda_{D'}(s,v^*) \geq \lambda_D(s,v^*)-1$. We will give some idea how to prove this under our matroid setting, but for now, we just suppose we can split off e,f edges of v^* such that $\lambda_{\mathcal{M}'}(s,v)=\lambda_{\mathcal{M}}(s,v)$ for every $v \in V-v^*$, and $\lambda_{\mathcal{M}'}(s,v^*) \geq \lambda_{\mathcal{M}}(s,v^*)-1$, where $\mathcal{M}'$ is the new matroid. If it is true, then k will be unchanged, because $\lambda_{\mathcal{M}}(s,v^*)$ was the minimum, so $k_{new}=\max\{\lambda_{\mathcal{M}'}(s,v) : v \in V\}=\max\{\lambda_{\mathcal{M}}(s,v) : v \in V-v^*, \lambda_{\mathcal{M}}(s,v^*)-1\}=k$. Let the new free edge $\overrightarrow{u_1u_2}$ be g.

By induction we know that there are k arc-disjoint s-arborescences $F_1,...,F_k$ and every $v \in V-v^*$ vertex is in at least $\lambda_{\mathcal{M}'}(s,v)=\lambda_{\mathcal{M}}(s,v)$ arborescences, and v^* is in at least $\lambda_{\mathcal{M}}(s,v^*)-1$.

If no F_i contains g (the new edge), then let $F_i' := F_i$ for every $i \in \{1,...,k\}$, they will only use the original edges.

If for an index i, F_i contains g (only one can contain it), then we have 2 cases.

Case 1.: v^* is not contained by F_i. Then let $F_i' := F_i - g + e + f$ and if $j \neq i$ $F_j' := F_j$. These arborescences are arc-disjoint because e,f edges were not in $\mathcal{M}'$. And they are arborescences, since we just "*split*" the edge g into two part.

Case 2.: v^* is contained by F_i, then there is a unique sv^* path P in F_i. Let $h := \overrightarrow{wv^*}$ be last edge of P.

If $g \in P$, then we throw out g, so we cannot reach u_2, but we add edges e,f, so every vertex can be reached in $V(F_i)$. In this case we go from u_1 to v^*, then to u_2, but in v^* there would be two incoming edges, so we remove h from F_i. Then $F_i' := F_i - g - h + e + f$ is an arborescence, and $F_j' := F_j$.

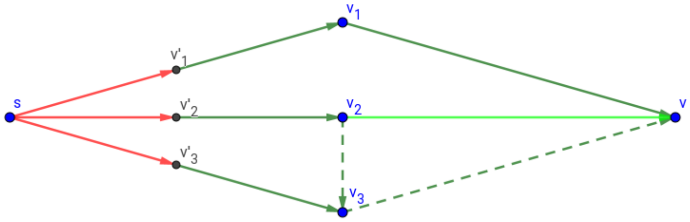
If $g \notin P$, then if we remove g from F_i, and add only f, then we can reach u_2 from v^*, but here this vertex has only one incoming edge (h), so we do not delete h. Then $F_i' := F_i - g + f$ is an arborescence, $F_j' := F_j$.

So now we have arborescences in $\mathcal{M}$. Every vertex, which is not v^* or s, is in the same arborescences, so they are contained by at least $\lambda_{\mathcal{M}}(s,v)$ arborescences.

If v^* was in at least $\lambda_{\mathcal{M}}(s, v^*)$ arborescences, then we are done.
Suppose v^* is contained by exactly $\lambda_{\mathcal{M}}(s, v^*)-1$ arborescences. Because $\lambda_{\mathcal{M}}(s, v^*)$ was the minimum, for every $v \in V - v^*$ there is an arborescence which contains v, but not v^*, since there are less arborescences which contain v^*, then v. Since the indegree of v^* is at least $\lambda_{\mathcal{M}}(s, v^*)$, there is an incoming arc which is not used by any F'_i (e or h), and let its tail vertex be u. Suppose $u \notin S$, then u is covered by $\lambda_{\mathcal{M}}(s, u)$ arborescences, and there is an arborescence F'_j, which does not contain v^*, so we can add to F'_j the edge $\overrightarrow{uv^*}$, and the new arborescence remains arc-disjoint from the others. The other case, when the only not contained, incoming edge of v^* is the edge $\overrightarrow{v'v^*}$ and $v' \in S$ (so the original root-edge was $\overrightarrow{sv^*}$). The arborescences do not have to cover the vertices of S. If there is an arborescence F'_j which contains v', then we only add this vertex to the arborescence, because we want to reach its "pair". v' has only 1 indegree and 1 outdegree, so $\overrightarrow{sv'} \in F'_j$, and $\overrightarrow{v'v^*} \in F'_j$, but it is a contradiction. So no F'_j contains v', but there is at least one, which does not contain v^*, and we can add to this arborescence the edges $\overrightarrow{sv'}$ and $\overrightarrow{v'v^*}$.
Then v^* will be contained by $\lambda_{\mathcal{M}}(s, v^*)$ arborescences and they will be arc-disjoint, so we are done.
In this proof we did not care about whether the packing of the arborescences is "independent" in the sense, if we take a vertex v, then the root-edges of the unique sv paths of the arborescences are independent. $\square$

It would be good, if the above defined $\lambda_{\mathcal{M}}(s, v)$ would be equal with $\min\{\varrho^r(X) : X \subseteq V, v \in X\}$, like in Menger's theorem. In our case $\varrho^r(X) = \varrho_D(X) \geq \lambda_D(s, v)$ where $v \in X \subseteq V$, and $\lambda_{\mathcal{M}}(s, v) \leq \lambda_D(s, v)$ since there cannot be more arc-disjoint path in D, and we get $\varrho^r(X) \geq \lambda_{\mathcal{M}}(s, v)$. If a set X is tight, so $\varrho^r(X) = \lambda_{\mathcal{M}}(s, v)$, then we saw that after the splitting the left side can decrease, and the right side also. However in the following example there can be a case, when only the right side decrease.
Let a submatroid of $\mathcal{M}$ be the following: s is the root, there are edges $\overrightarrow{sv_1}$, $\overrightarrow{sv_2}$ and $\overrightarrow{sv_3}$, we cut them into two parts and $v'_1, v'_2, v'_3 \in S$. There is a vertex v and the free edges are $\overrightarrow{v_2v_3}$, $\overrightarrow{v_1v}$ and $\overrightarrow{v_3v}$, furthermore, v can be reached from s only on these edges in D. Assume that $\overrightarrow{sv'_1}$ and $\overrightarrow{sv'_2}$ are dependent, but $\overrightarrow{sv'_3}$ is independent from both of them. Then sv'_1v_1v and sv'_3v_3v are arc-disjoint sv paths (this is the maximum number of arc-disjoint paths) and their root-edges are independent, so $\lambda_{\mathcal{M}}(s, v) = 2$, and $\varrho^r(\{v\}) = 2$, so $X = \{v\}$ is tight. But, if we split off $\overrightarrow{v_2v_3}$ and $\overrightarrow{v_3v}$ edges, then $\overrightarrow{v_2v}$ is the new free edge, but since the two arc-disjoint paths are sv'_2v_2v and sv'_1v_1v and their root-edges are dependent,

$\lambda_{\mathcal{M}'}(s,v)=1$, but every set X which contains v, but disjoint from s and S, has indegree 2, so $\varrho^r_{\mathcal{M}'}(X)\geq 2$. Of course, if we care about the sets which are not disjoint from S, then for $X=\{v'_1,v_1,v'_2,v_2,v\}$ $\varrho^r_{\mathcal{M}'}(X)=1$.



We need something similar to Menger's theorem to be able to split off edges, but we cannot ignore the root-edges. However, we created the set S, because if we split off a root-edge, the new root-edge has to defined in the matroid $\mathcal{M}'_s$, and this is the hard part. For the future plans we show the main ideas of the original proof ([1]) and where are the hard parts.

Let $\beta_D(X):=\{\varrho_D(X),\delta_D(X)\}$, $R(X)=\max\{r(u,v):X \textit{ separates } u \textit{ and } v\}$ where $r(u,v)$ is a symmetric non-negative function. Specially, $r(u,v):=\min\{k,\lambda_D(u,v)\}$ where k is a lower bound of $\lambda_D(x,y)$ for every x,y which indegree and outdegree are not equal. If the graph is Eulerian, then $r(u,v)=\lambda_D(u,v)$ (this is symmetric), this is why we added new edges to the graph in the proof. Here we call a set X tight, if $\beta_D(X)=R(X)$, so X separates a u and a v vertex and $\beta_D(X)=\lambda_D(u,v)$.

When we split off edges $\overrightarrow{u_1v}$ and $\overrightarrow{vu_2}$, then the indegree and the outdegree can also decrease by one, but only if $u_1,u_2\in X$, so it can be shown that an edge pair can be split off iff there is no tight set which contains both u_1 and u_2. The idea is that we suppose $\lambda_{\mathcal{M}}(s,v):=\min\{\varrho^r(X):X\subseteq V\cup S, v\in X\}$, and we call $X\subseteq V\cup S$ tight, if $\lambda_{\mathcal{M}}(X):=\max\{\lambda_{\mathcal{M}}(s,v):v\in X\}$ is equal with $\varrho^r(X)$. However, this is maybe not the same λ as in the previos proof. Also, the definiton of β_D is not the same in $\mathcal{M}$ because $\delta_D(X)=\varrho_D(V+s-X)$, but in our assumptions we only care about the sets which does not contain s, because we want only the $\lambda_{\mathcal{M}}(s,v)$ values.

For the future plans, we would like to prove this split off theorem with matroids, or do another matroid-based proof for the main theorem.

References

[1] Jørgen Bang-Jensen, András Frank, and Bill Jackson. Preserving and increasing local edge-connectivity in mixed graphs. *SIAM Journal on Discrete Mathematics*, 8(2):155–178, 1995.

[2] Romain Chapoullié and Zoltán Szigeti. On packing time-respecting arborescences. *Discrete Optimization*, 45:100702, 2022.