Cost Sharing Methods for the Traveling Salesman Game

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1 Introduction

In cooperative game theory, rather than competing, players are able to form coalitions. They are willing to pay for some kind of service. The cost of serving any subset of players is known. In the traveling salesman game, we have a set of players and a distinguished depot. The cost of each coalition is defined as the weight of the minimum-weight Hamiltonian cycle on the given subset of nodes, including the depot. Computing the optimal cost share is strongly NP-hard for the traveling salesman game. My goal in this report is to show—with the results of [1]—how to derive a good cost share for this game.

2 The Traveling Salesman Game and the OCSP

Given a set of players $V = \{1, \ldots, n\}$ and a cost function $c : 2^V \to \mathbb{R}$, which satisfies $c(\emptyset) = 0$. A cost share is a $w_1, \ldots, w_n$ allocation among the players ($\forall i \in [n] : w_i \in \mathbb{R}$). Ideally, one would like to obtain an *efficient* cost share: $\sum_{i=1}^{n} w_i = c(V)$. The cost shares that are efficient and satisfy the *core* property:

$$\sum_{i \in S} w_i \leq c(S) \quad \forall S \subseteq V$$

form the core. Unfortunately, for some problems, the core is empty. Therefore, we can also consider a version in which we only require

$$\sum_{i=1}^{n} w_i = \gamma \cdot c(V)$$

for some $\gamma \in [0, 1]$; this is called γ-*budget balanced.* Maximizing γ, while maintaining the core property is the *optimal cost share problem* (OCSP). It can be formulated with this Linear Program:

$$\max \left\{ \sum_{i \in V} w_i : \sum_{i \in S} w_i \leq c(S), S \subseteq V \right\} \tag{1}$$

In the *traveling salesman game*, we are given a complete graph $G = (V \cup \{0\}, E)$, with c_e costs for each edge. The vertex 0 is the depot. For every subset S of V, that is not the empty set, the cost $c(S)$ is defined as the cost of the minimum-weight Hamiltonian cycle for the induced subgraph on $S \cup \{0\}$. Thus, it can be calculated by solving the following Integer Program:

$$c(S) = \min \left\{ \begin{array}{l} cx : \displaystyle\sum_{e \in \delta(0)} x_e = 2, \\ \displaystyle\sum_{e \in \delta(i)} x_e = 2y_i(S), \quad i \in V, \\ \displaystyle\sum_{e \in \delta(R)} x_e \geq 2y_i(S), \quad R \subseteq V, i \in R, \\ x \in \mathbb{Z}_+^{|E|} \end{array} \right\}. \tag{2}$$

Where $\delta(H)$ is the set of edges that have one endpoint in H and one outside of H. The notation $\delta(\{v\}) = \delta(v)$ is used. $y(S)$ is the incidence vector with the property $y_i(S) = 1$ if $i \in S$, and $y_i(S) = 0$ otherwise. This program can be written in a compact form:

$$c(S) = \min\{cx : Ax \geq By(S) + d, x \in \mathbb{Z}_+^m\}. \tag{3}$$

In particular for the ground set V, we have:

$$\min\{cx : Ax \geq B\mathbf{1} + d, x \in \mathbb{Z}_+^m\}, \tag{4}$$

where $\mathbf{1}$ is the all-one vector, and $m = |E|$. Here A contains the characteristic vector for all the constraints, namely: the degree equations for all the vertices, and the subtour elimination constraints. Note that d cannot be the all-zero vector, because of the degree constraint for the depot, that is the inequality is not homogeneous.

3 Assignable Inequalities and Constraint Generation

We want to show that there is a strong connection between the computation of a good cost share and the computation of lower bounds for $c(V)$, i.e., the TSP. Row generation is one of the best approaches for calculating these lower bounds considering the amount of research in polyhedral combinatorics. Therefore, we will use valid inequalities as cutting planes.

First, let Q^{xy} denote the solution pairs for every Integer Program from 3, where $n = |V|$, that is

$$Q^{xy} := \{x \in \mathbb{Z}_+^m, y \in \{0,1\}^n : Ax \geq By + d, \ y \neq \mathbf{0}\}.$$

The convex hull of the integer solutions of 4 forms the polyhedron:

$$P_I^x := \text{conv}\{x \in \mathbb{Z}_+^m : Ax \geq B\mathbf{1} + d\}.$$

The set of feasible solutions for the LP relaxation forms the well-known subtour elimination polytope:

$$P^x = \left\{ x \in [0,1]^{|E|} : \sum_{e \in \delta(i)} x_e = 2, \quad i \in V \cup \{0\}, \quad \sum_{e \in \delta(R)} x_e \geq 2, \quad R \subseteq V, R \neq \emptyset \right\}.$$

Also, let

$$P_I^{xy} := \operatorname{conv} Q^{xy}.$$

For deriving the cost share for each vertex, we need inequalities that have a certain property, i.e., they are assignable. Using the definitions and results from [1]:

Definition 1. An inequality $\alpha x \geq \beta$ which is valid for P_I^x is said to be *assignable* if there exists an inequality $\alpha x \geq \gamma y$ which is valid for P_I^{xy}, and such that: $\sum_{i \in V} \gamma_i = \beta$.

For example, the degree constraint $x(\delta(v)) \geq 2$ is obviously valid for the P_I^x for any $v \in V$, where $x(\delta(v)) = \sum_{e \in \delta(v)} x_e$. It can be reformulated as $x(\delta(v)) \geq \gamma y$ with $\gamma_u = 2$ if $u = v$, and 0 otherwise; this is valid for P_I^{xy}, because for every problem where $v \in S$, the sum of the x values on the edges that are incident to v has to be 2.

In [1] they proved that if we have a set of assignable inequalities $Dx \geq f$ which corresponds to a collection $Dx \geq Ey$ homogeneous inequalities valid for P_I^{xy}, with $E\mathbf{1} = f$ and a separation algorithm for $Dx \geq Ey$, then a cost share can be found in polynomial time, with value

$$\min\{cx : Dx \geq f\}.$$

This means that $w(V)$ can be calculated by solving an LP with constraint generation. The individual shares can be calculated using the following formula:

$$w_i = \sum_{e_{hi} \neq 0} e_{hi} \pi_h^*,$$

where e_{hi} is the element of E in the h-th row and i-th column, and π_h^* is the optimal dual solution for the h-th inequality.

Applying these results for the traveling salesman game we already saw that the degree equations are assignable except for the depot. Also, the subtour elimination constraints are assignable for subsets not containing the depot. For the latter constraints, the corresponding homogeneous inequalities are $x(\delta(R)) \geq \gamma y$, where $\gamma \mathbf{1} = 2$. They need to be valid for P_I^{xy}, so we can choose any $v \in R$, and set $\gamma = 2$ on v and 0 otherwise. We could also distribute 2 evenly on R. This means that we can manipulate the cost share without decreasing $w(V)$ or violating the core property.

4 Blossom Inequalities

Since we can separate for the subtour inequalities in polynomial time, we can already calculate a good cost share optimizing on P^x. Improving, we may consider another well-known class of facet-inducing inequalities for the TSP, the so-called *clique tree* inequalities [2]. According to [1], these are also assignable if we consider their *weakened* version. These inequalities are not used directly as ordinary blossom inequalities, but in a weakened form that preserves assignability. Thus, they improve the lower bound while still allowing the dual solution to be translated into player-wise cost shares. Unfortunately, there is no known separation algorithm for these general inequalities, but in a special case of the blossom inequalities, we can separate.

Definition 2. A blossom is defined by a handle $H \subset V$ and a set of teeth $M \subseteq \delta(H)$, where $|M| \geq 3$ is odd and M is a matching.

Theorem 3. *For the traveling salesman game, the weakened blossom inequalities:*

$$x(\delta(H)) + x(\delta(M)) + \left\lfloor \frac{|M|}{2} \right\rfloor x(\delta(0)) \geq 4|M| \tag{5}$$

$$x(\delta(H)) + x(\delta(M)) + \left\lfloor \frac{|M|}{2} - 1 \right\rfloor x(\delta(0)) \geq 4|M| - 2 \tag{6}$$

are assignable if the depot does not lie in a tooth and lies in a tooth, respectively.

With these special inequalities, we can increase the optimum of P^x, and the additional cost share for each player can be derived because changing the right-hand side of 5 to $2\sum_{v \in M'} y_v$ is valid for P_I^{xy}, where M' is the $2|M|$ endpoints of M, hence $\gamma_v = 2$ if $v \in M'$ and 0 otherwise. Similarly for 6.

For the separation, we may use that 5 and 6 can be reformulated as follows:

$$x(\delta(H) - M) + \left(|M| \frac{x(\delta(0))}{2} - x(M) \right) \geq \frac{x(\delta(0))}{2}, \tag{7}$$

since $\forall e = uv \in M : x(\delta(u, v)) = 4 - 2x_e$, if the depot is not in e, and $x(\delta(e)) = x(\delta(0)) + 2 - 2x_e$ if the depot is either u or v. This suggests that for a suitable auxiliary graph, the separation can be done similarly as for the non-weakened inequalities [5], solving minimal T-cut problems.

In particular, using the notations and formulation from [4], let G' be the following graph. From G, remove the edges for which $x_e = 0$. Subdivide each of the remaining edges with two new vertices: for an edge $e = uv$, let these be v_e' and v_e''. The weights of the edges uv_e' and vv_e'' shall remain x_e, while the weight of the edge $v_e'v_e''$ shall be $\frac{x(\delta(0))}{2} - x_e$. Let T consist of the set of these new vertices. So we only change the weights of the edges defined by two new vertices, and also according to 7 we will check if the minimal T-cut is less than $\frac{x(\delta(0))}{2}$.
Now, let $x \in \mathbb{R}^m$ violate the blossom inequality for the pair H, M. Let S be the vertices in G' corresponding to H, the new vertices spanned by H, and the new vertices on each edge of M that are closer to H. Then clearly the cut S is a T-cut and it has value less than $\frac{x(\delta(0))}{2}$ as x was violating.
For the other direction, let S be a minimal T-cut in G' that has value less than $\frac{x(\delta(0))}{2}$. We want to find a pair H, M such that the corresponding blossom inequality is violated by x. For this, we simply reverse the previous method, and we get that H, M satisfy the definition 2.

5 Results

The implementation included both subtour separation and the weakened blossom inequalities using the LEMON Graph Library. For comparison, I compared the resulting cost shares with the cost share derived from the Shapley values [6]. Note that it is really slow to calculate the Shapley value, as it solves many Integer Programs, because it is an expected value that I approximated by averaging over 100 randomly sampled permutations. Thus, I used a small graph with 19 vertices corresponding to Hungarian cities. In this instance, $c(V) = 1671$, the core is non-empty, and the algorithm finds an optimal cost share (that is in the core). The results show that the total allocated cost is the same in all three cases, but the individual allocations differ substantially. Also, if we evenly distribute the dual values for the subtour inequality, we get a completely different cost share.

City Name	**Shapley Value**	**Initial LP Cost**	**Distribution**
Veszprem	52.06	131	72.29
Tatabanya	66.38	63	66.29
Szombathely	55.63	105	105.00
Szolnok	115.93	133	111.54
Szekszard	66.28	146	108.63
Szekesfehervar	40.10	47	56.29
Szeged	131.51	107	114.54
Salgotarjan	131.81	149	152.29
Pecs	59.85	52	74.13
Nyiregyhaza	198.39	151	111.79
Miskolc	111.81	71	74.29
Kecskemet	47.90	44	62.54
Kaposvar	33.27	79	104.13
Gyor	60.92	87	90.29
Eger	102.37	68	71.29
Debrecen	161.00	46	91.79
Bekescsaba	199.34	123	131.54
Budapest	36.45	69	72.29
Total	1671	1671	1671

Table 1: Comparison of Cost Sharing Methods: Shapley Values vs. the LP method described in this report, where the Initial LP Cost is the cost share that we get if we always increase the cost of the first city (in a certain order) that is not on the side of the cut containing the depot, and the Distribution is when we evenly distribute between the cities that are not on the side containing the depot. The depot is Zalaegerszeg. The edge weights of the graph are the distances in kilometres between the cities.

6 Conclusion

The report illustrates how assignable inequalities can be used to derive cost shares for the traveling salesman game from dual solutions of LP relaxations. The computational example shows that different valid assignments of the same inequalities can lead to substantially different individual cost shares, even when the total allocation remains unchanged. This highlights that, besides budget balance and core membership, the choice of assignment rule has a significant effect on the resulting allocation.

References

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