Cost Sharing Methods for the Traveling Salesman Game

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Main question: how can dual information from TSP relaxations be turned into player-wise cost shares?

From last semester: cost sharing games

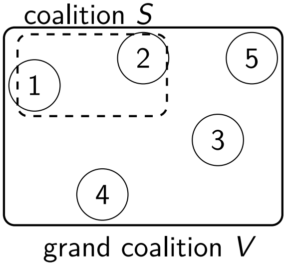
Basic setting

- Players: $V = \{1, \ldots, n\}$.
- Coalition cost: $c : 2^V \to \mathbb{R}$, with $c(\emptyset) = 0$.
- Cost share: $w_1, \ldots, w_n$.

Core property

$$\sum_{i \in S} w_i \leq c(S) \qquad \forall S \subseteq V.$$

No coalition should pay more inside the grand coalition than it would pay alone.



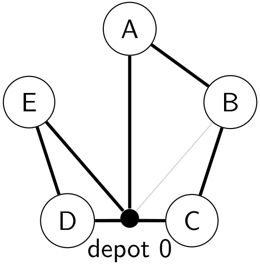
OCSP

Maximize the allocated total while preserving the core inequalities:

$$\max \left\{ \sum_{i \in V} w_i : \sum_{i \in S} w_i \leq c(S) \right\}.$$

The Traveling Salesman Game

- Complete graph on $V \cup \{0\}$.
- Vertex 0 is the depot.
- For a coalition $S \subseteq V$, $c(S)$ is the minimum TSP tour through $S \cup \{0\}$.



Coalition cost

$$
\begin{aligned}
c(S) = \min\{cx : \; & x(\delta(0)) = 2, \\
& x(\delta(i)) = 2y_i(S), \\
& x(\delta(R)) \geq 2y_i(S), \\
& x \in \mathbb{Z}_+^{|E|}\}.
\end{aligned}
$$

Computational difficulty

Computing the exact optimal cost share is strongly NP-hard for the TSP game.

Main theoretical tool: assignable inequalities

Idea

A valid inequality for the TSP polytope is useful for cost sharing if its right-hand side can be distributed among the players.

Definition

An inequality $\alpha x \geq \beta$ valid for the integer TSP polyhedron is *assignable* if there exists a valid inequality

$$\alpha x \geq \gamma y$$

for the mixed (x, y) formulation such that

$$\sum_{i \in V} \gamma_i = \beta.$$

valid TSP inequality → dual value π_h^* $\xrightarrow{\gamma}$ player shares

How the LP produces a cost share

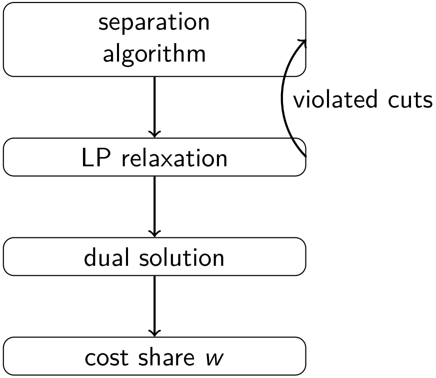
LP with generated inequalities

$$\min\{cx : Dx \geq f\}.$$

The rows of D are assignable inequalities, with a corresponding assignment matrix E satisfying $E\mathbf{1} = f$.

Cost share from the dual

$$w_i = \sum_{h : e_{hi} \neq 0} e_{hi} \pi^*_h.$$



The total allocation equals the lower bound obtained from these inequalities.

First assignable family: subtour inequalities

Subtour elimination

$$x(\delta(R)) \geq 2 \qquad \emptyset \neq R \subseteq V.$$

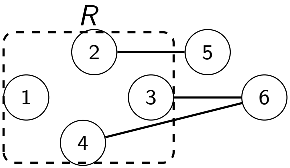
For P_I^{xy} this can be made homogeneous as

$$x(\delta(R)) \geq \gamma y, \qquad \sum_i \gamma_i = 2.$$

Important freedom

The value 2 can be assigned to one vertex of R, or distributed among the vertices of R.

- same LP lower bound;
- same total allocation;
- potentially very different individual cost shares.



Main extension: weakened blossom inequalities

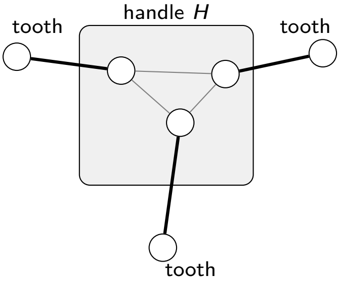
Blossom structure

A handle $H \subset V$ and a set of teeth $M \subseteq \delta(H)$, where M is a matching and $|M| \geq 3$ is odd.

$$x(\delta(H)) + x(\delta(M)) + \left\lfloor \frac{|M|}{2} \right\rfloor x(\delta(0)) \geq 4|M|,$$

$$x(\delta(H)) + x(\delta(M)) + \left\lfloor \frac{|M|}{2} - 1 \right\rfloor x(\delta(0)) \geq 4|M| - 2.$$

These are weakened forms of blossom inequalities. The weakening is the price paid for assignability.

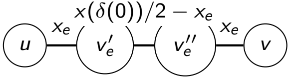


Role in the project

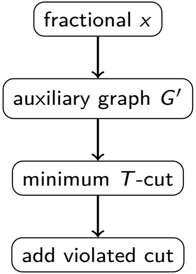
They strengthen the subtour LP while preserving a valid conversion from dual values to cost shares.

Separation idea for the blossom cuts

1. Start from the current fractional solution x.
2. Build an auxiliary graph G'.
3. Subdivide each positive edge with two new vertices.
4. Define a special set T of the new vertices.
5. Find a minimum T-cut.



subdivided edge $e = uv$



Violation test

A cut of value below $x(\delta(0))/2$ corresponds to a violated weakened blossom inequality.

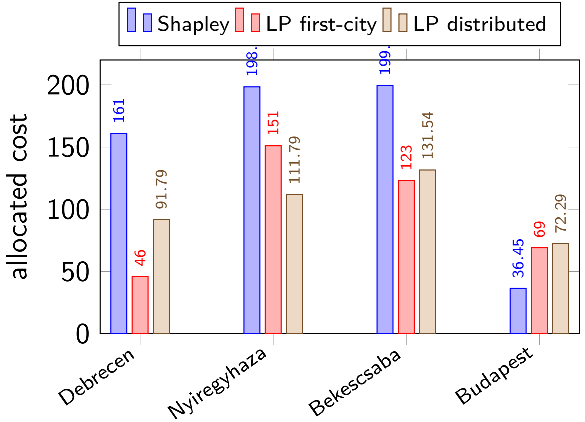
Implementation and computational comparison

- Implemented subtour separation and weakened blossom cuts.
- Used the LEMON Graph Library.
- Test instance: 19 Hungarian cities.
- Depot: Zalaegerszeg.
- Edge weights: road distances in kilometres.

Instance result

$$c(V) = 1671,$$

with a non-empty core; the LP method found a core allocation.



The total is 1671 in all three shown allocations, but the individual shares are very different.

What the results show

Theoretical conclusion

Assignable inequalities connect two tasks:

- strengthening TSP lower bounds;
- constructing valid cost shares.

Computational conclusion

The same LP optimum can produce substantially different player allocations, depending on how the assignable inequalities are distributed.

Main message for the presentation

Budget balance and core membership do not determine a unique "fair" allocation. The assignment rule is an additional modelling choice.

- Shapley is general and intuitive.
- LP-based allocation is core-oriented.
- Assignment choices remain important.

References

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Thank you for your attention!

AI

I used AI for code generation.