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Applied Mathematics MSc

THE ACTIVITY OF THE STOCHASTIC CHIP-FIRING GAME

Individual Project 1
Report

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Introduction

In [2], we have already seen that the stochastic chip-firing game can be viewed as an aperiodic, irreducible Markov-chain on the recurrent states of chip configurations, and therefore it has a unique stationary distribution. It has also been proved that a recurrent chip-configuration of the stochastic chip-firing game on a path graph with n vertices and $m = n - 1$ chips can be viewed the following way: we take an arbitrary orientation of the edges, then the number of chips a vertex has equals its indegree in the orientation. An inner vertex can fire, if the edge to its left is oriented from left to right, and the edge to its right is oriented from right to left. The first vertex is active (=can fire) if its indegree is 1, meaning that the first edge is oriented from right to left. The last vertex is active if the last edge is oriented from left to right. This gives a bijection between recurrent chip-configurations and orientations of the path graph. We therefore have 2^m recurrent states in the Markov chain of the game above. For simplicity, we denote the states by m-long binary strings, 0 denoting an edge that is oriented from left to right, while 1 denotes an edge that is oriented from right to left. When an inner vertex fires, the two incident edges change from 01 to 10. When the first vertex fires, the first edge changes from 1 to 0, and when the last vertex fires, the last edge switches from 0 to 1. The activity of the stochastic chip-firing game is the expected value of the average number of active vertices. In the following, I present the relationship of the stochastic chip-firing game and a well-known model. Then I use the results of that model to prove a formula regarding the activity of the path graph.

Proof of the activity formula

Theorem 1 *The activity of the stochastic chip-firing game on a path with m edges and m chips is*

$$\frac{m^2 + 3m - 2}{4m - 2}.$$

Next, as it appears in [1], we consider the one-dimensional Totally Asymmetric Simple Exclusion Process (TASEP), in which a path graph with N sites (one-dimensional lattice of N sites) is given, and each site is either occupied by a particle or it is empty. Every particle has its own exponential clock of rate 1 and it jumps to the left when the clock rings if the site to the left is empty (for every site except for the last). If the leftmost site is occupied, the particle in that site can leave the system, meaning that a particle that is in the last site is removed from the system with rate β. Similarly, if the rightmost site is empty, a particle can enter the system with rate α. In our case, $\alpha = \beta = 1$. We can denote a site by 0 if it is empty, and 1 if it is occupied.

The relation between the two processes (chip-firing on a path graph with m edges and TASEP with m sites) can be noticed easily. When an inner vertex (say i-th) fires in the chip-firing game, it means that the neighboring two edges ($(i-1)$-th and i-th) change from 01 to 10 and this is equivalent to the following: in the one-dimensional TASEP, a particle at the i-th site jumps to the $(i-1)$-th. So in both binary strings, the same two bits change from the same state to the same state.

The one-dimensional TASEP is a continuous time Markov-chain (CTMC), while its jump chain is the chip-firing game. Let π_m and λ_m denote the stationary distribution of the m-vertex chip-firing game's Markov chain and the CTMC of the m-site TASEP, respectively. Let $a(\tau)$ denote the number of active vertices in state τ in the chip-firing game, this is the same as the number of possible changes to state τ in the TASEP. Let $M = 2^m$, so for a TASEP with m sites, the stationary distribution is M-dimensional. From Section 3.5. in [3], we know that a CTMC with stationary distribution $\lambda(x)$ and exit rates (total rate at which the process leaves a state) $q(x)$, the stationary distribution of the jump chain is proportional to $\lambda(x)q(x)$. It is easy to see that in our case, the exit rate of the TASEP in state τ is $a(\tau)$. So $\pi_m(\tau) = \frac{a(\tau)\cdot\lambda_m(\tau)}{\mathbf{E}_{\lambda_m}(a)}$. So if we want to find the expected value of the activity in the chip-firing game, we need $\mathbf{E}_{\pi_m}(a)$. It is easy to see that

$$\mathbf{E}_{\pi_m}(a) = \sum_\tau a(\tau)\cdot\pi_m(\tau) = \sum_\tau a(\tau)\cdot\frac{a(\tau)\cdot\lambda_m(\tau)}{\mathbf{E}_{\lambda_m}(a)} = \frac{\mathbf{E}_{\lambda_m}(a^2)}{\mathbf{E}_{\lambda_m}(a)}.$$

Let us examine a in more detail. To that end, we define indicator variables for each of the m sites. Since a particle can only leave the system if the first site is occupied (it is 1), for a state τ, for the corresponding indicator variable I_0 we have: $I_0(\tau) = \tau_1$. Similarly, for the exit we have $I_m(\tau) = 1 - \tau_m$. For an inner site, a possible move is when the corresponding digits of the binary string are 01, so for $1 \le i \le m-1$ we have $I_i(\tau) = (1-\tau_i)\tau_{i+1}$. So

$$a(\tau) = I_0(\tau) + I_n(\tau) + \sum_{i=1}^{m-1} I_i(\tau).$$

Based on Chapter 2 in [1], we know that the probability of a state τ in stationary, denoted by $P_m(\tau)$, can be obtained by:

$$P_m(\tau) = \frac{f_m(\tau_1,\tau_2,\ldots,\tau_m)}{Z_m},$$

where

$$f_m(\tau) = \langle W|\prod_{i=1}^{m}(\tau_i D + (1-\tau_i)E)|V\rangle$$

and

$$Z_m = \sum_{\tau_1=0,1}\cdots\sum_{\tau_m=0,1} f_m(\tau_1,\tau_2,\ldots,\tau_m).$$

In the above formula, D and E denote square matrices, while $\langle W|$ and $|V\rangle$ are vectors that satisfy the following:

$$ED = E + D,$$
$$E|V\rangle = |V\rangle,$$
$$\langle W|D = \langle W|.$$

Using the result in Chapter 10 of [1], we know that $Z_m = C_{m+1}$, where C_k denotes the k-th Catalan number.

In the formula of f_m, one can see that it is given by a matrix product of m matrices, each being D or E. The i-th matrix is D if the i-th site is occupied, and E if it is empty.

Using this, if we define $C = D + E$, then it is easy to see that

$$Z_m = \langle W|C^m|V\rangle.$$

Furthermore,

$$E_{\lambda_m}(I_0) = \frac{\langle W|DC^{m-1}|V\rangle}{Z_m} = \frac{Z_{m-1}}{Z_m}.$$

Similarly,

$$E_{\lambda_m}(I_m) = \frac{\langle W|C^{m-1}E|V\rangle}{Z_m} = \frac{Z_{m-1}}{Z_m}.$$

Now, for the I_i-s corresponding to the inner vertices:

$$E_{\lambda_m}(I_i) = \frac{\langle W|C^{i-1}EDC^{m-i-1}|V\rangle}{Z_m} = \frac{Z_{m-1}}{Z_m}.$$

In the last equality, we used that $ED = E + D$.

Based on the three expected values, we can see easily that

$$\mathbf{E}_{\lambda_m}(a) = (m+1)\frac{Z_{m-1}}{Z_m} = (m+1)\frac{C_m}{C_{m+1}} = (m+1)\frac{\frac{1}{m+1}\binom{2m}{m}}{\frac{1}{m+2}\binom{2m+2}{m+1}} =$$

$$= (m+1)\frac{\frac{1}{m+1}\frac{(2m)!}{m!m!}}{\frac{1}{m+2}\frac{(2m+2)!}{(m+1)!(m+1)!}} = (m+2)\frac{(m+1)^2}{(2m+1)(2m+2)} = \frac{(m+1)(m+2)}{2(2m+1)}.$$

From now on, only $\mathbf{E}_{\lambda_m}(a^2)$ is left to calculate.

$$a^2 = \sum_{i=0}^{m} I_i + 2\sum_{0\le i\le j\le m} I_iI_j.$$

Now we turn our focus to $\mathbf{E}_{\lambda_m}(I_iI_j)$. If i and j are adjacent sites, then I_i and I_j cannot be both 1 at the same time. We have four cases:

- if $i = 0$ and $2 \le j \le m-1$, then

$$\mathbf{E}_{\lambda_m}(I_iI_j) = \frac{\langle W|DC^{j-2}EDC^{m-j-1}|V\rangle}{Z_m} = \frac{\langle W|C^{j-2}CC^{m-j-1}|V\rangle}{Z_m} = \frac{\langle W|C^{m-2}|V\rangle}{Z_m} = \frac{Z_{m-2}}{Z_m}$$

- if $1 \le i \le m-2$ and $j = m$, then

$$\mathbf{E}_{\lambda_m}(I_iI_j) = \frac{\langle W|C^{i-1}EDC^{m-i-2}E|V\rangle}{Z_m} = \frac{\langle W|C^{i-1}CC^{m-i-2}|V\rangle}{Z_m} = \frac{\langle W|C^{m-2}|V\rangle}{Z_m} = \frac{Z_{m-2}}{Z_m}$$

- if $i = 0$ and $j = m$, then

$$\mathbf{E}_{\lambda_m}(I_iI_j) = \frac{\langle W|DC^{m-2}E|V\rangle}{Z_m} = \frac{\langle W|C^{m-2}|V\rangle}{Z_m} = \frac{Z_{m-2}}{Z_m}$$

- If $1 \le i < j \le m-1$ and $i < j-1$, then

$$\mathbb{E}_{\lambda_m}(I_iI_j) = \frac{\langle W \mid C^{i-1}EDC^{j-i-2}EDC^{m-j-1} \mid V\rangle}{Z_m} = \frac{\langle W \mid C^{i-1}CC^{j-i-2}CC^{m-j-1} \mid V\rangle}{Z_m} =$$

$$= \frac{\langle W \mid C^{m-2} \mid V\rangle}{Z_m} = \frac{Z_{m-2}}{Z_m}.$$

So all four cases end up giving $\frac{Z_{m-2}}{Z_m}$. Now we have to count the number of pairs. We can choose any two of the $(m+1)$ possibilities, except for the adjacent ones, which gives $\binom{m+1}{2} - m = \frac{m(m-1)}{2}$. So

$$a^2 = \sum_{i=0}^{m} I_i + 2 \sum_{0 \le i \le j \le m} I_i I_j = (m+1)\frac{Z_{m-1}}{Z_m} + m(m-1)\frac{Z_{m-2}}{Z_m}$$

.

Finally, we can obtain $\frac{\mathbf{E}_{\lambda_m}(a^2)}{\mathbf{E}_{\lambda_m}(a)}$.

$$\frac{\mathbf{E}_{\lambda_m}(a^2)}{\mathbf{E}_{\lambda_m}(a)} = \frac{(m+1)\frac{Z_{m-1}}{Z_m} + m(m-1)\frac{Z_{m-2}}{Z_m}}{(m+1)\frac{Z_{m-1}}{Z_m}}.$$

If we multiply both the numerator and the denominator by Z_m and use that

$$\frac{Z_{m-2}}{Z_{m-1}} = \frac{C_{m-1}}{C_m} = \frac{m+1}{2(2m-1)},$$

we get that

$$\frac{\mathbf{E}_{\lambda_m}(a^2)}{\mathbf{E}_{\lambda_m}(a)} = \frac{(m+1)Z_{m-1} + m(m-1)Z_{m-2}}{(m+1)Z_{m-1}} = 1 + \frac{m(m-1)}{m+1}\frac{m+1}{2(2m-1)} = 1 + \frac{m(m-1)}{2(2m-1)} = \frac{m^2+3m-2}{4m-2}.$$

This is the formula we wanted to obtain.

Further plans

Now that the activity formula for the two "ends" (the path and the star) of the poset on trees with n nodes defined in my thesis is known, the next goal is to prove that the activity is increasing on this poset. This poset is based on the lexicographic order of the degree sequence, arranged in decreasing order. In this poset, for example, $P_n < S_n$ holds, since the degree sequence of P_n is $2, 2, 1, 1, ..., 1$ and the degree sequence of S_n is $n-1, 1, 1, ..., 1$.

Bibliography

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