The activity of the stochastic chip-firing game

Proof of the activity formula for path graphs

Domonkos Koleszár
Supervisor: Lilla Tóthmérész

Eötvös Loránd University

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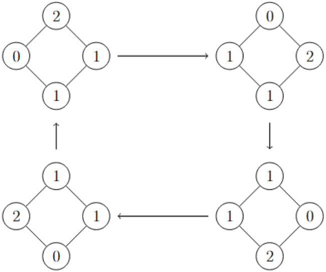
Summary

1. Background
2. The path formula
3. Connection with TASEP
4. Matrix product calculation
5. Further plans

Introduction

- Dynamics of the game
- Parallel chip-firing
- Asynchronous chip-firing
- Stochastic chip-firing



Activity

Activity of the stochastic chip-firing game

For a graph G and initial chip-distribution x, the activity is

$$A_G(x) = \lim_{k\to\infty} \frac{1}{k}\mathbb{E}(A_k(x)),$$

where $A_k(x)$ denotes the total number of active vertices during the first k steps.

- The activity on a tree with $n-1$ chips depends only on the tree, not on the initial chip configuration.

Stochastic chip-firing game on trees with n vertices and $n-1$ chips

Star

The activity of the stochastic chip-firing game with $n-1$ chips on a star with n vertices is always

$$A_{S_n} = \frac{n^2 - n + 2}{2n}.$$

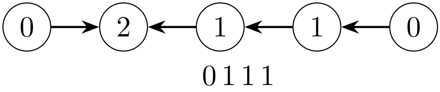
Conjecture for path graphs

The activity of the stochastic chip-firing game with $n-1$ chips on a path with n vertices is always $A_{P_n} = \frac{n^2+n-4}{4n-6}$.

Chip-firing on a path

- Recurrent chip configurations: arbitrary orientation of the edges of the tree, for each vertex: number of chips = indegree $\Longrightarrow$ 2^m recurrent states for the path graph with m chips
- Aperiodic, irreducible Markov-chain on the recurrent states of chip configurations $\Longrightarrow$ stationary vector π
- Activity $= \pi \cdot a$, where a denotes the number of active vertices for the 2^m states

- Binary encoding of orientations: 0=left-to-right, 1=right-to-left.



Main theorem

Theorem

The activity of the stochastic chip-firing game on a path with m edges and m chips is

$$A_{P_{m+1}} = \frac{m^2 + 3m - 2}{4m - 2}.$$

- In terms of $n = m + 1$ vertices, the formula is

$$A_{P_n} = \frac{n^2 + n - 4}{4n - 6}.$$

Previous attempts

- Activity = sum of the activities of the vertices
- Conjectures for the activity of the nodes
  - Inner vertices: $\frac{C_m+(m-1)C_{m-1}}{\binom{2m}{m}}$, where C_m denotes the m-th Catalan-number
  - The two vertices at the end of the path graph: $\frac{\binom{2m-2}{m}\cdot\frac{m+4}{m+1}}{\binom{2m}{n}}$
- In stationary distribution, every vertex has the same probability of firing (so $\frac{1}{n}$)

Binary states and possible firings

For a state $\tau = (\tau_1, \ldots, \tau_m) \in \{0,1\}^m$:

- first vertex active: $\tau_1 = 1$;
- last vertex active: $\tau_m = 0$;
- inner vertex $i+1$ active: the adjacent bits are 01.

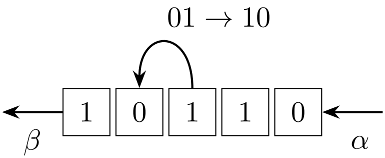
$$\boxed{0 \,|\, 1} \longrightarrow \boxed{1 \,|\, 0}$$

inner firing

So the number of active vertices in state τ is the number of possible local changes.

TASEP with open boundaries

- Totally Asymmetric Simple Exclusion Process on m sites.
- Each site is either empty (0) or occupied (1).
- A particle can jump left if the site to its left is empty.
- Open boundaries with rates $\alpha = \beta = 1$.



Key observation

The chip-firing Markov chain is the jump chain of this continuous-time TASEP.

From CTMC to jump chain

Let λ_m be the stationary distribution of TASEP and π_m the stationary distribution of the chip-firing jump chain.

Jump-chain relation

If $q(\tau)$ is the exit rate from state τ, then

$$\pi_m(\tau) = \frac{q(\tau)\lambda_m(\tau)}{\mathbb{E}_{\lambda_m}(q)}.$$

In this model,

$$q(\tau) = a(\tau),$$

where $a(\tau)$ is the number of active vertices in the chip-firing state. Therefore

$$\mathbb{E}_{\pi_m}(a) = \frac{\mathbb{E}_{\lambda_m}(a^2)}{\mathbb{E}_{\lambda_m}(a)}.$$

Matrix product form

The stationary weight of a TASEP state is described by

$$P_m(\tau) = \frac{f_m(\tau)}{Z_m}, \qquad f_m(\tau) = \langle W| \prod_{i=1}^{m} (\tau_i D + (1-\tau_i)E) |V\rangle.$$

The matrices and vectors satisfy

$$ED = E + D, \qquad E|V\rangle = |V\rangle, \qquad \langle W|D = \langle W|.$$

Normalization

With $C = D + E$,

$$Z_m = \langle W|C^m|V\rangle = C_{m+1},$$

where C_k is the k-th Catalan number.

Indicator variables

Define indicators for the possible moves:

$$I_0(\tau) = \tau_1, \qquad I_m(\tau) = 1 - \tau_m,$$

$$I_i(\tau) = (1 - \tau_i)\tau_{i+1} \quad (1 \le i \le m - 1).$$

Then

$$a(\tau) = \sum_{i=0}^{m} I_i(\tau).$$

Using the matrix relation $ED = E + D$, each move has the same expectation:

$$\mathbb{E}_{\lambda_m}(I_i) = \frac{Z_{m-1}}{Z_m} \quad \text{for every } i = 0, \dots, m.$$

Thus

$$\mathbb{E}_{\lambda_m}(a) = (m + 1)\frac{Z_{m-1}}{Z_m} = \frac{(m + 1)(m + 2)}{2(2m + 1)}.$$

Second moment

For adjacent possible moves, $I_i I_j = 0$. For every non-adjacent pair,

$$\mathbb{E}_{\lambda_m}(I_i I_j) = \frac{Z_{m-2}}{Z_m}.$$

The number of non-adjacent unordered pairs among $m+1$ possible moves is

$$\binom{m+1}{2} - m = \frac{m(m-1)}{2}.$$

Therefore

$$\mathbb{E}_{\lambda_m}(a^2) = (m+1)\frac{Z_{m-1}}{Z_m} + m(m-1)\frac{Z_{m-2}}{Z_m}.$$

Finishing the proof

$$\frac{\mathbb{E}_{\lambda_m}(a^2)}{\mathbb{E}_{\lambda_m}(a)} = \frac{(m+1)Z_{m-1} + m(m-1)Z_{m-2}}{(m+1)Z_{m-1}}.$$

Since $Z_m = C_{m+1}$,

$$\frac{Z_{m-2}}{Z_{m-1}} = \frac{C_{m-1}}{C_m} = \frac{m+1}{2(2m-1)}.$$

Hence

$$\frac{\mathbb{E}_{\lambda_m}(a^2)}{\mathbb{E}_{\lambda_m}(a)} = 1 + \frac{m(m-1)}{2(2m-1)} = \frac{m^2+3m-2}{4m-2}.$$

This proves the path activity formula.

Consequences and further plans

- The activity formula is now known for both extremal examples in the degree-sequence poset:

$$P_n \quad \text{and} \quad S_n.$$

- The next goal is to prove that the stochastic activity is increasing on the poset of trees ordered by decreasing degree sequence.

- In particular, this would imply that among trees on n vertices:

$$A_{P_n} \leq A_T \leq A_{S_n}.$$

Thank you for your kind attention!