

# Optimization problems in temporal graphs

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Project Work II.

2026. 06. 04.

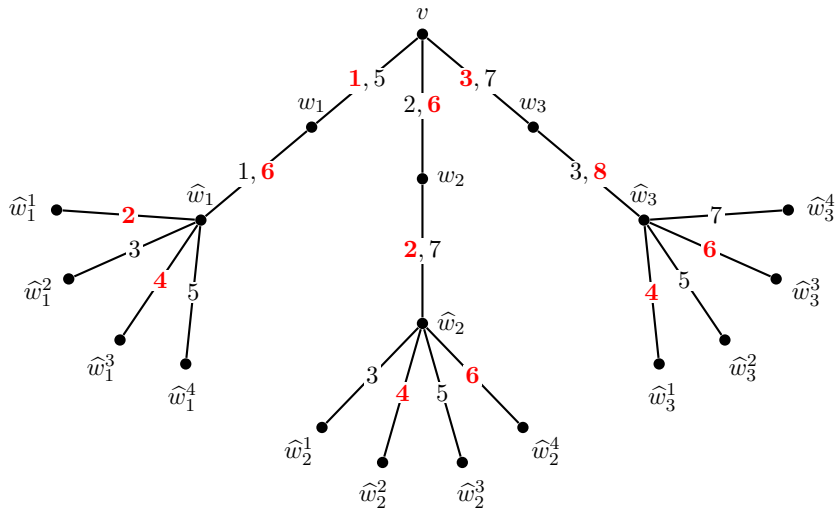
# Definition: Temporal graphs

- $\mathcal{G} = (G, \lambda)$ ,  $G = (V, E)$  is an underlying (static) graph and  $\lambda : E \rightarrow 2^{\mathbb{N}} \setminus \{\emptyset\}$  is a time-labeling function.
- An edge  $e \in E$  is active at time  $\tau \in \mathbb{N}$  if  $\tau \in \lambda(e)$ .
- If  $e$  is active at time  $\tau$ , then we call the pair  $(e, \tau)$  a time edge.
- The lifetime  $\mathcal{T}(\mathcal{G})$  is the largest time-label assigned to an edge:  $\mathcal{T}(\mathcal{G}) = \max \bigcup_{e \in E} \lambda(e)$ .

# Definition: $\Delta$ -matching

- Two time edges  $(e, \tau)$  and  $(e', \tau')$  are  $\Delta$ -independent if the edges  $e, e'$  do not share an endpoint or  $|\tau - \tau'| \geq \Delta$ .
- A  $\Delta$ -matching of a temporal graph  $\mathcal{G}$  is a set  $M$  of pairwise  $\Delta$ -independent time edges of  $\mathcal{G}$ .
- In the maximum  $\Delta$ -matching problem, we want to find a  $\Delta$ -matching with the largest possible cardinality.

# Example of a $\Delta$ -matching



# Definition: $\gamma$ -matching

- For an edge  $e \in E$  and a time tick  $\tau \in \mathbb{Z}_{>0}$  such that  $[\tau, \tau + \gamma - 1] \subseteq \lambda(e)$ , define  $(e, \tau)_\gamma = \{(e, \tau') : \tau' \in [\tau, \tau + \gamma - 1]\}$ . This is the  $\gamma$ -edge of  $e$  starting at time  $\tau$ .
- Two  $\gamma$ -edges  $(e, \tau)_\gamma$  and  $(e', \tau')_\gamma$  are  $\gamma$ -independent if the edges  $e, e'$  do not share an endpoint or the intervals  $[\tau, \tau + \gamma - 1]$  and  $[\tau', \tau' + \gamma - 1]$  are disjoint.
- A  $\gamma$ -matching of  $\mathcal{G}$  is a set of pairwise  $\gamma$ -independent  $\gamma$ -edges.
- In the maximum  $\gamma$ -matching problem, we want to find a  $\gamma$ -matching with the largest possible cardinality.

# Definition: $d$ -matching

- The input is a bipartite graph  $G = (S, T, E)$  with  $S = \{s_1, \dots, s_n\}$  and a positive integer  $d$ .
- The goal is to find a subset  $M \subseteq E$  of maximum cardinality such that every vertex in  $S$  is incident with at most one edge of  $M$ , and whenever  $s_i t, s_j t \in M$  for some  $t \in T$  with  $i \neq j$ , we have  $|i - j| \geq d$ .
- Alternatively,  $M \subseteq E$  is a  $d$ -matching if and only if, after restricting  $M$  to any interval of at most  $d$  consecutive vertices of  $S$ , the remaining edges form a matching.

## Theorem

*The maximum  $\Delta$ -matching problem is NP-hard even if  $\Delta = 2$ , the underlying graph of the input temporal graph is a tree and every edge appears at most twice.*

## Theorem

*The maximum  $\Delta$ -matching problem can be solved in polynomial time if the underlying graph  $G = (V, E)$  of the input temporal graph  $\mathcal{G} = (G, \lambda)$  is a tree, and every edge appears at most once.*

## Theorem

*The maximum  $d$ -matching problem can be solved in polynomial time if the input graph  $G$  is a tree.*

## Theorem

*The maximum  $\gamma$ -matching problem is NP-hard, even if the underlying graph of the temporal graph is a tree.*

## Lemma

*The maximum  $d$ -matching problem is a special case of the maximum  $\Delta$ -matching problem, in which every edge appears exactly once.*

## Theorem

*For every  $\gamma \geq 2$ , the maximum  $\gamma$ -matching problem is NP-hard, even if the underlying graph is a tree and each edge admits at most two  $\gamma$ -edges.*

## Theorem

*Consider the maximum  $\Delta$ -matching problem, where the underlying graph  $G = (V, E)$  of the input temporal graph  $\mathcal{G} = (G, \lambda)$  is a tree. If there exists an optimal solution  $M^*$  for which the sets  $M_v^* = \{(e, t) \in M^* \mid e \text{ is incident to } v\}$  are of size at most  $K$  for every  $v \in V$ , then the problem can be solved in  $O(n^2(K\mathcal{T}^2)^K)$  time.*

## Theorem

*Consider the maximum  $\Delta$ -matching problem on a temporal graph  $\mathcal{G} = (G, \lambda)$  whose underlying graph  $G = (V, E)$  is a tree, and let  $K \in \mathbb{Z}_{>0}$  be given. For every vertex  $v \in V$ , let  $A_v = \bigcup_{e \ni v} \lambda(e)$  be the set of time ticks at which at least one edge incident to  $v$  is active. Let  $B = \max_{v \in V} |A_v|$ . Suppose that there exists an optimal solution  $M^*$  such that, for every vertex  $v \in V$ , the time edges of  $M^*$  incident to  $v$  use at most  $K$  time ticks. Then an optimal solution can be found in  $O((2eB)^K K^2 n)$  time, where  $n = |V|$  and  $e$  denotes Euler's number.*

## Corollary

*Consider the maximum  $\gamma$ -matching problem, where the underlying graph  $G = (V, E)$  of the input temporal graph  $\mathcal{G} = (G, \lambda)$  is a tree, and let  $K \in \mathbb{Z}_{>0}$  be given. Let  $B_\gamma = \max_{v \in V} |\{(e, \tau)_\gamma : e \in E, e \ni v, [\tau, \tau + \gamma - 1] \subseteq \lambda(e)\}|$ . If there exists an optimal solution  $M^*$  for which the sets  $M_v^* = \{(e, \tau)_\gamma \in M^* : e \text{ is incident to } v\}$  have size at most  $K$  for every  $v \in V$ , then an optimal solution can be found in  $O((2eB_\gamma)^K K^2 n)$  time, where  $n = |V|$  and  $e$  denotes Euler's number.*

## Theorem

*Consider the maximum  $\Delta$ -matching problem, where the underlying graph  $G = (V, E)$  of the input temporal graph  $\mathcal{G} = (G, \lambda)$  is a tree. Let  $N$  denote the size of the input ( $N = O(|E|\mathcal{T}) = O(|V|\mathcal{T})$ ). If  $\sum_{u \in \Gamma(v)} |\lambda(uv)| = O(\log N)$  for every  $v \in V$ , where  $\Gamma(v)$  denotes the set of neighbors of  $v$ , then the problem can be solved in polynomial time.*

## Theorem

*For every fixed constant  $c$ , the maximum  $\Delta$ -matching problem can be solved in polynomial time on instances whose underlying graph  $G = (V, E)$  is a tree and which satisfy  $\sum_{u \in N_G(v)} |\lambda(uv)| \leq c \log L$  for every  $v \in V$ , where  $L = \sum_{e \in E} |\lambda(e)|$  and  $N_G(v)$  denotes the set of neighbors of  $v$  in  $G$ .*

## Corollary




*For every fixed constant  $c$ , the maximum  $\gamma$ -matching problem can be solved in polynomial time on instances whose underlying graph  $G = (V, E)$  is a tree and which satisfy  $|\{(e, \tau)_\gamma : e \in E, e \ni v, [\tau, \tau + \gamma - 1] \subseteq \lambda(e)\}| \leq c \log L_\gamma$  for every  $v \in V$ , where  $L_\gamma$  denotes the total number of  $\gamma$ -edges in the instance.*

## Theorem

*For every  $0 < \varepsilon < 1$ , the maximum  $\Delta$ -matching problem admits an  $O(\mathcal{T}n^2(1/\varepsilon)^{3/\varepsilon})$ -time  $(1 - \varepsilon)$ -approximation algorithm if the underlying graph  $G$  of the input temporal graph  $\mathcal{G}$  is a tree.*

## Theorem

*Let  $\mathcal{G} = (G, \lambda)$  be a temporal graph whose underlying graph  $G = (V, E)$  is a tree, and let  $n = |V|$ . Write  $L = \sum_{e \in E} |\lambda(e)|$  and  $B = \max_{v \in V} |A_v|$ , where  $A_v = \bigcup_{e \ni v} \lambda(e)$ . For every  $0 < \varepsilon < 1$ , the maximum  $\Delta$ -matching problem on  $\mathcal{G}$  admits a  $(1 - \varepsilon)$ -approximation algorithm with running time  $O(L^2(2eB)^{\lceil 1/\varepsilon \rceil} \lceil 1/\varepsilon \rceil^2 n)$ . In particular, since  $B \leq L$ , the running time is  $L^{O(1/\varepsilon)}(n)$ , and the problem admits a PTAS.*

-  George B. Mertzios, Hendrik Molter, Rolf Niedermeier, Viktor Zamaraev, and Philipp Zschoche. „*Computing maximum matchings in temporal graphs*”. In: *Journal of Computer and System Sciences* 137 (2023), pp. 1–19.
-  Péter Madarasi. „*Matchings under distance constraints I.*” In: *Annals of Operations Research* 305 (2021), pp. 137–161.
-  Péter Madarasi. „*Matchings under distance constraints II.*” In: *Annals of Operations Research* 332 (2024), pp. 303–327.

# Where did I use AI?

I used ChatGPT to translate my Hungarian sentences to English and to generate figures.

Thank you for your attention!