Numerical Discretization and Structure Preservation

Scaled Korteweg-de Vries (KdV) Equation

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The Scaled KdV Equation & Hamiltonian Structure

- **Solitary Waves:** Arise from a perfect balance between nonlinear convection and linear dispersion, traveling at a constant velocity without changing shape.
- **Governing Equation & Boundary Conditions:**

$$u_t + (u^2)_x + 6u_{xxx} = 0, \quad u(t, -L/2) = u_x(t, -L/2) = u(t, L/2) = u_x(t, L/2) = 0$$

- **Hamiltonian System:** The equation can be written as:

$$u_t = J\frac{\delta H}{\delta u}$$

where $J = -\partial_x$ is a skew-adjoint differential operator, and the Hamiltonian functional is:

$$H[u] = \int \left(\frac{u^3}{3} - 3u_x^2\right) dx$$

- **Corresponding Variational Derivative:**

$$\frac{\delta H}{\delta u} = u^2 + 6u_{xx}$$

Hence,

$$u_t = -\partial_x(u^2 + 6u_{xx}) = -(u^2)_x - 6u_{xxx}$$

Motivation for Structure Preservation

- **The Problem:** Naive discretizations often introduce artificial dissipation or dispersion, gradually destroying qualitative wave features (e.g., shape preservation, phase shifts).
- **The Goal:** Construct a numerical scheme that preserves the underlying geometric structure.
- **Core Strategy:**
  1. Preserve the skew-symmetry of the operator J.
  2. Transform the continuous PDE into a discrete Hamiltonian ODE system.
  3. Apply geometric time integrators for temporal evolution.

1.1.1 Spatial Discretization

- **Domain & Grid:** Bounded domain $x \in [-L/2, L/2]$ with periodic boundary conditions. Approximated by discrete vector $U(t) = [u_1(t), \ldots, u_N(t)]^T$.
- **Preserving Skew-Symmetry:** The continuous operator $J = -\partial_x$ is approximated by:

  $$J_h = -D_1$$

  where D_1 is the central finite difference matrix.
- **Key Property:** D_1 is skew-symmetric ($D_1^T = -D_1$). Hence, $J_h^T = -J_h$, successfully preserving the geometric structure of the continuous operator.

1.1.1 Discrete Hamiltonian System

- **Discrete Hamiltonian:** Replacing the integral with a Riemann sum:

$$H_h(U) = h\sum_{i=1}^{N}\left(\frac{U_i^3}{3} - 3(D_1 U)_i^2\right)$$

- **Semi-discrete ODE:** This yields a finite-dimensional Hamiltonian ODE:

$$\dot{U} = J_h \nabla H_h(U)$$

- **Implementation:** The discrete gradient is evaluated consistently with the continuous variational derivative:

$$\nabla H_h(U) = (U_1^2, \ldots, U_N^2)^T + 6D_2 U$$

1.1.2 Temporal Discretization: The Implicit Midpoint Rule

- **Symplectic Integrator:** We apply the Implicit Midpoint Rule to integrate the ODE:

$$\frac{U^{n+1}-U^{n}}{\Delta t}=J_h\nabla H_h\left(\frac{U^{n+1}+U^{n}}{2}\right)$$

- **Advantage:** Keeps the discrete energy error bounded without secular drift.
- **Nonlinear Solver:** Since it is implicit, the actual unknown U^{n+1} is obtained by solving the residual equation:

$$F(U^{n+1})=U^{n+1}-U^{n}-\Delta t J_h\nabla H_h\left(\frac{U^{n+1}+U^{n}}{2}\right)=0$$

  using the **Newton-Raphson iteration** method until a strict tolerance is met.

1.2 Reflective Boundary Conditions

- **The Challenge:** In a finite domain with natural/reflective boundaries, using periodic central difference matrices destroys the skew-symmetric structure.
- **The Solution: Strang Operator Splitting Method**
- We decompose the original equation into two sub-problems:
  1. Nonlinear convection part: $u_t + (u^2)_x = 0$
  2. Linear dispersion part: $u_t + 6u_{xxx} = 0$

1.2 Strang Splitting Cycle

To advance the system by one full time step Δt, we use a three-step cycle:

1. **Nonlinear convection (Half-step $\Delta t/2$):** Solved using the Implicit Midpoint Rule.
2. **Linear dispersion (Full-step Δt):** Advanced using the Crank-Nicolson method with a specially constructed skew-symmetric operator L_{mat}, which preserves the discrete L^2 norm.
3. **Nonlinear convection (Half-step $\Delta t/2$):** A final half-time step is applied to complete the splitting cycle.

2.1 Single Soliton Propagation (Initial State)

- Verification of structure-preserving properties.
- Propagation of a single wave under periodic BCs.

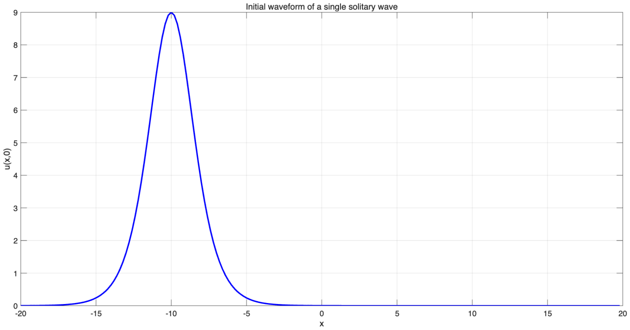

Figure: Initial waveform at $t = 0$.

2.1 Single Soliton Propagation (Evolution)

- Energy error remains bounded.
- Non-dissipative behavior confirms well-preserved solution structure.

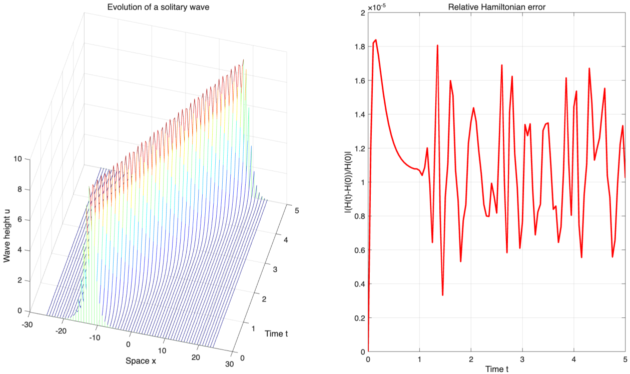

Figure: Spatiotemporal evolution and Hamiltonian error.

2.1 Convergence Study

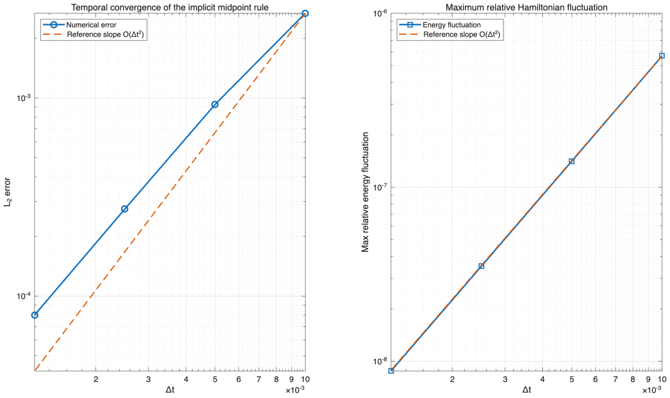

Figure: Confirming second-order temporal convergence (L_2 and Hamiltonian error).

2.2 Two-Soliton Elastic Collision (Contour)

- Nearly elastic interaction.
- Taller wave travels faster and overtakes the shorter one.
- Result: A visible phase shift after interaction.

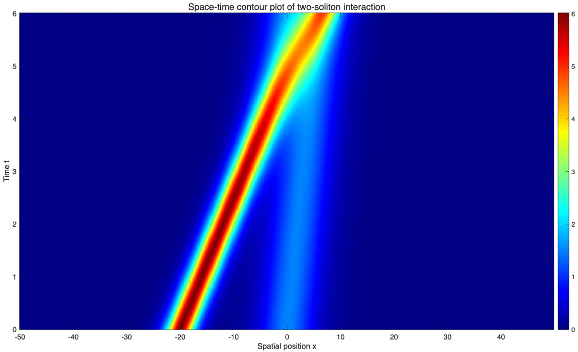

Figure: Space-time contour of the interaction process.

2.2 Two-Soliton Elastic Collision (Snapshots)

- Waves recover original shapes and amplitudes after separation.
- Demonstrates the high quality of the numerical scheme.

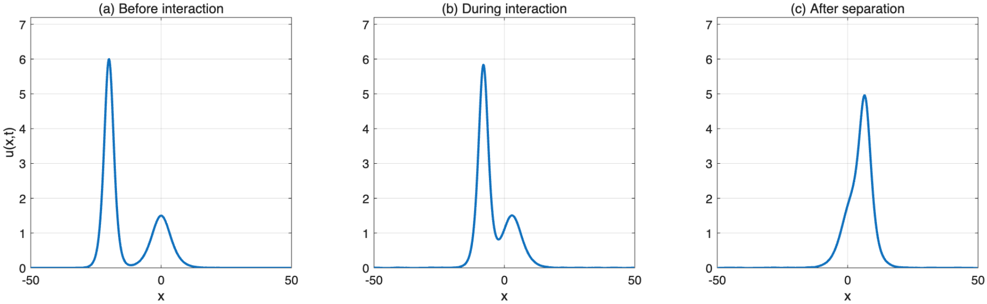

Figure: Snapshots: Before, During, and After collision.

2.3 Boundary Reflection Dynamics

- Interaction with a reflective boundary.
- Wave energy propagates backward as high-frequency dispersive ripples.
- Splitting method captures stable reflection dynamics.

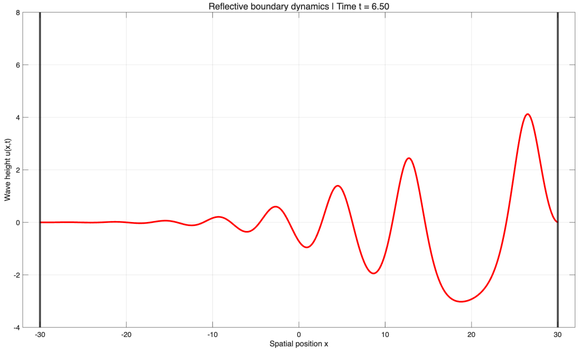

Figure: Wave behavior near a natural boundary.

AI Usage Statement

- During the preparation of this project, Artificial Intelligence tools were utilized to assist with the following tasks:
    - **Literature Translation:** Assisting in the translation of reference materials to ensure accurate academic comprehension.
    - **Language Polishing:** Refining and polishing sentences for better clarity, grammar, and academic tone in the final report.
    - **Code Debugging:** Identifying syntax errors and providing troubleshooting suggestions when the simulation code encountered execution issues.

Thank You!