Epidemics on Hypergraphs 2

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This report is a continuation of my previous semester's work entitled "Epidemics on Hypergraphs", in which I introduce the hypergraph model for the spread of diseases to be discussed, as well as briefly refer to some results of my research regarding the bifurcation landscape of the resulting ODE systems in the case of hypergraphs on 3 nodes. In this update I first restate the fundamentals of the model for context and then dive deeper into the aforementioned bifurcation analyses. Finally, I introduce a sub-model which, although imposes heavy limitations on the structure of the underlying hypergraph, is capable of handling hypergraphs of arbitrary size while keeping ODE systems at a tractable 2 equations. The general framework of the hypergraph models presented in this report is based on an unpublished draft of the second edition of [2]. Certain relevant concepts also appear in the published first edition [1] for the case of graphs.

For the purpose of this work we consider SIS dynamics on hypergraphs with hyperedges of order at most three. The dynamics considered are the following: infection occurs across standard edges at a rate of $\tau > 0$ and across hyperedges of order three at a rate of $\beta > 0$, as long as all but one node within the hyperedge are infected. The recovery of infected nodes is assumed to occur at a rate of $\gamma > 0$. Within the bottom-up model, we derive differential equations for the probability of a given state having a given status. Let $\langle I_i \rangle(t)$ and $\langle S_i \rangle(t)$ denote, respectively, the probabilities that node i is infected and susceptible at time t. After introducing similar variables for the joint probability of multiple nodes having certain statuses at time t, we apply a closure at the level of singles using the independence conditions $\langle A_i B_j \rangle = \langle A_i \rangle \langle B_j \rangle$ and $\langle A_i B_j C_k \rangle = \langle A_i \rangle \langle B_j \rangle \langle C_k \rangle$, where $A, B, C \in \{S, I\}$ and i, j, k are the indices of the relevant nodes. Lastly, we use the relationship $\langle S_i \rangle = 1 - \langle I_i \rangle$ to arrive at the simplest form of the ODE systems. For example, in the case of $N = 3$ nodes, if the hypergraph contains two edges, $\{1, 2\}$ and $\{2, 3\}$, alongside the hyperedge of order three, then we have the system

$$\begin{aligned}
\dot{\langle I_1 \rangle} &= \tau(1 - \langle I_1 \rangle)\langle I_2 \rangle + \beta(1 - \langle I_1 \rangle)\langle I_2 \rangle\langle I_3 \rangle - \gamma\langle I_1 \rangle, \\
\dot{\langle I_2 \rangle} &= \tau(1 - \langle I_2 \rangle)(\langle I_1 \rangle + \langle I_3 \rangle) + \beta(1 - \langle I_2 \rangle)\langle I_1 \rangle\langle I_3 \rangle - \gamma\langle I_2 \rangle, \\
\dot{\langle I_3 \rangle} &= \tau(1 - \langle I_3 \rangle)\langle I_2 \rangle + \beta(1 - \langle I_3 \rangle)\langle I_1 \rangle\langle I_2 \rangle - \gamma\langle I_3 \rangle.
\end{aligned}$$

In case of identical initial conditions at nodes 1 and 3, the temporal evolution of the corresponding variables will be identical, that is we have $\langle I_1 \rangle(t) = \langle I_3 \rangle(t) = x(t)$ for all t. Thus, using the notation $y(t) = \langle I_2 \rangle(t)$, the system can be reduced to

$$\dot{x} = \tau(1 - x)y + \beta(1 - x)xy - \gamma x, \tag{1a}$$

$$\dot{y} = 2\tau(1 - y)x + \beta(1 - y)x^2 - \gamma y. \tag{1b}$$

In case the required equality of initial conditions is not met, the solution to the reduced system can be considered to be an approximation of the original variables. If the hypergraph contains only one edge, namely $\{1,2\}$, then the system takes the form

$$\dot{x} = \tau(1-x)x + \beta(1-x)xy - \gamma x, \tag{2a}$$

$$\dot{y} = \beta(1-y)x^2 - \gamma y. \tag{2b}$$

We now turn to the analysis of the above two systems. Our goal is to characterize the bifurcation diagram of each model, whereby the number and stability of the steady states can be determined for any combination of the parameters. Where possible, exact analytical solutions are given. For problems that are analytically intractable, we resort to numerical approximations.

1 One edge

We begin with the analysis of system (2). For a steady state, we require

$$0 = x\left(\tau(1-x) + \beta(1-x)y - \gamma\right),$$
$$0 = \beta(1-y)x^2 - \gamma y.$$

To determine the stability of the trivial, disease-free steady, we calculate the Jacobian of the right hand side at the origin. Observing that

$$J(0,0) = \begin{pmatrix} \tau-\gamma & 0 \\ 0 & -\gamma \end{pmatrix},$$

we have that the disease-free steady state is stable for $\tau < \gamma$ and unstable for $\tau > \gamma$.

In order to characterize the other equilibria, we turn to the Lagrange multiplier method for computing fold bifurcations as described in [3]. Essentially, we are searching for parameter combinations, for which the nullclines of system (2) are tangential to one another. We let $\gamma = 1$ and consider the altered system

$$\dot{x} = G(x,y;\beta) - \tau,$$
$$\dot{y} = F(x,y;\beta),$$

where $G(x,y,\beta) = \frac{1}{1-x} - \beta y$, which is obtained by dividing equation (2a) by $-x(1-x)$, and $F(x,y,\beta) = \beta(1-y)x^2 - y$. Note that the above system is not orbitally equivalent to the original, however nonzero equilibria are left unchanged. Furthermore, the appropriate scaling of the second equation (2b) does not change the tangency condition at equilibria and can thus be neglected.

We now fix the infection parameter β and look for a solution to the constrained optimization problem

$$F(x,y;\beta) = 0,$$
$$\nabla G(x,y;\beta) = \lambda \nabla F(x,y;\beta).$$

Writing out the system we are searching for $x, y \in (0,1)$ and $\lambda \in \mathbb{R}$ such that

$$\beta(1-y)x^2 - y = 0,$$
$$\frac{1}{(1-x)^2} = 2\lambda\beta(1-y)x,$$
$$\beta = \lambda(\beta x^2 + 1).$$

Expressing

$$y = y(x) = \frac{\beta x^2}{1+\beta x^2} \quad \text{and} \quad \lambda = \frac{\beta}{1+\beta x^2}$$

leads to the quartic equation in x

$$p(x) = \beta^2 x^4 - 2\beta^2 x^3 + (2\beta + 4\beta^2)x^2 - 2\beta^2 x + 1 = 0.$$

Observing that $p''(x) > 0$ for all x, we conclude that there exist at most two real roots. The solutions of this equation can be computed numerically and the corresponding values of τ can be determined by setting $\tau = \tau(x) = G(x, y(x); \beta)$. Treating β as a variable and adding the condition

$$p'(x) = 4\beta^2 x^3 - 6\beta^2 x^2 + (4\beta + 8\beta^2)x - 2\beta^2 = 0$$

leads us to finding a cusp bifurcation at $\beta = \beta^* = 2.1306516116$, $\tau = \tau^* = 1.0828781651$. Accordingly, the system exhibits two fold bifurcations for

$\beta > \beta^*$ and none for $\beta < \beta^*$.

Since $\tau(0) = \tau'(0) = 1$, forward type transcritical bifurcation occurs at $\tau = 1$ for all β. We note that for $\gamma \neq 1$, the same holds at $\tau = \gamma$.

We now shift our focus to the number and stability of the steady states. It can be shown that the stability of a steady state (x, y) is determined solely by the slope of the curve for τ: It is stable if $\tau'(x) > 0$ and unstable if $\tau'(x) < 0$. As a result, we have four cases, as shown and numbered in Figure 1.

Region 1: The only steady state is the trivial, disease-free one, which is globally stable.

Region 2: There are two steady states. The disease-free steady state is unstable and the other one is stable.

Region 3: Apart from the unstable trivial steady-state, we have three further steady states, two of which are stable, the other is unstable. That is, bistability occurs between to endemic steady states.

Region 4: There are three steady states. The disease-free steady state is stable together with one of the endemic steady states, i.e. bistability occurs. The other steady state is unstable.

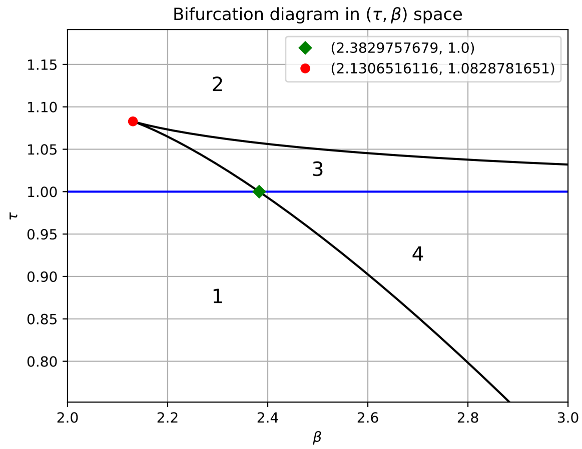
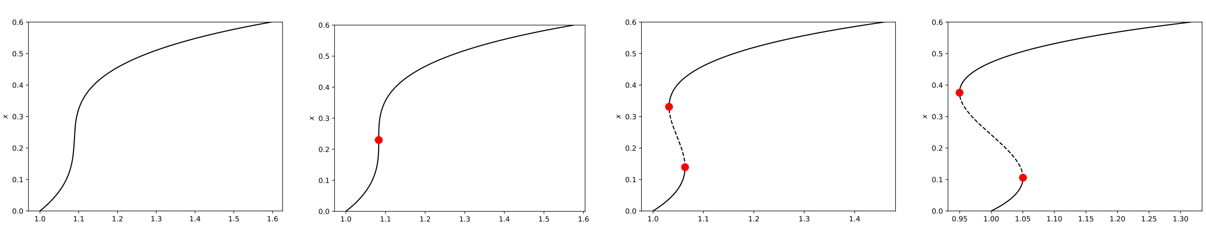


Figure 1: Bifurcation diagram of system (2) for $\gamma = 1$. The fold, cusp and transcritical bifurcations are illustrated by the two black curves, the red dot and the blue line, respectively. The parameter plane is divided into areas numbered one through four, based on the number and stability of the steady states.

To further clarify the given characterization of the steady state structure of the model, we provide bifurcation diagrams in (τ, x) space for representative values of β in Figure 2.



(a) $\beta = 2.1$ (b) $\beta = \beta^*$ (c) $\beta = 2.3$ (d) $\beta = 2.5$

Figure 2: Bifurcation diagrams of system (2) in (τ, x) space for various values of β. Regions 1 and 2 of Figure 1 are represented on all diagrams. Sections of region 3 can be seen on the two bottom diagrams, and region 4 appears only in the diagram for $\beta = 2.5$.

2 Two edges

We now continue with system (1). For a steady state, we require

$$0 = \tau(1-x)y + \beta(1-x)xy - \gamma x,$$
$$0 = 2\tau(1-y)x + \beta(1-y)x^2 - \gamma y.$$

First, we consider the trivial, disease-free steady state. Calculating the Jacobian of the right hand side, we find that

$$J(0,0) = \begin{pmatrix} -\gamma & \tau \\ 2\tau & -\gamma \end{pmatrix},$$

meaning that the disease-free steady state is stable for $\tau < \sqrt{2}\gamma$ and unstable for $\tau > \sqrt{2}\gamma$. Thus, the transcritical bifurcation in this model occurs at $\tau = \sqrt{2}\gamma$. Depending on the value of β, this can be either forward or backward type. This dependence is uncovered after determining the other branch of the bifurcation curve.

Expressing y from both equations for the steady states as

$$y = \frac{\gamma x}{(1-x)(\tau+\beta x)} = \frac{2\tau x + \beta x^2}{2\tau x + \beta x^2 + \gamma},$$

we obtain a quadratic equation for τ in terms of x and the remaining parameters:

$$A\tau^2 + B\tau + C = 0,$$

where $A = 2(1-x)$, $B = x\left(3\beta(1-x) - 2\gamma\right)$ and $C = \beta^2x^2(1-x) - (\beta\gamma x^2 + \gamma^2)$. The discriminant can be simplified to the form

$$D = (\beta x(1-x) - 2\gamma x)^2 + 8\gamma^2(1-x),$$

which is positive for all $x \in (0,1)$, thus both solutions

$$\tau_{\pm}(x) = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

to the quadratic are real. However, the curve τ_- is negative for $x \in (0,1)$. This can be proven by showing that for all $x \in (0,1)$, $B > 0$ implies $4AC < 0$. Therefore, the curve of nontrivial steady states is $\tau = \tau_+$.

Using implicit differentiation, the derivative of τ with respect to x is

$$\tau'(x) = -\frac{A'\tau^2(x) + B'\tau(x) + C'}{2A\tau(x) + B},$$

where $A' = -2$, $B' = 3\beta(1-2x) - 2\gamma$ and $C' = 2\beta^2x(1-x) - \beta^2x^2 - 2\beta\gamma x$. Using $\tau(0) = \sqrt{2}\gamma$ to solve the equation $\tau'(0) = 0$, we have that forward type transcritical bifurcation occurs for $\beta < \beta^* = \frac{\sqrt{2}+2}{3}\gamma$ and backward type transcritical bifurcation occurs for $\beta > \beta^*$. In order to find the fold bifurcation in the latter case, we must solve the equation $A'\tau^2(x) + B'\tau(x) + C' = 0$. Multiplying by $(1-x)$ and adding $A\tau^2(x) + B\tau(x) + C = 0$, the τ^2 term is eliminated and we get that

$$\begin{aligned}\tau(x) &= -\frac{(1-x)C' + C}{(1-x)B' + B} \\ &= -\frac{2\beta^2x(1-x)^2 - 2\beta\gamma x + \beta\gamma x^2 - \gamma^2}{3\beta(1-x)^2 - 2\gamma},\end{aligned}$$

where B' and C' denote the derivatives of B and C with respect to x, respectively. Solving the above equation for x numerically and substituting back into $\tau(x)$ yields the fold bifurcation $\tau = \tau_0$.

Summarizing our findings, we have the following cases regarding the number and stability of the steady states, which can also be read from Figure 3. Forward type transcritical bifurcation occurs for $\beta < \beta^*$. In this case, we have the following two scenarios.

- If $\tau < \sqrt{2}\gamma$, then the disease-free one is the only steady state and it is globally stable.
- If $\tau > \sqrt{2}\gamma$, then the disease-free steady state becomes unstable, and a further, stable steady state appears.

In the case of backward type transcritical bifurcation, i.e. for $\beta > \beta^*$, we have the following three scenarios.

- If $\tau < \tau_0$, then the only steady state is the disease-free one and it is globally stable.
- If $\tau_0 < \tau < \sqrt{2}\gamma$, then we have three steady states. The disease-free steady state is stable together with one of the endemic ones, i.e. bistability occurs. The other endemic steady state is unstable.
- If $\tau > \sqrt{2}\gamma$, then there are two steady states. The trivial one is unstable, while the nontrivial one is stable.

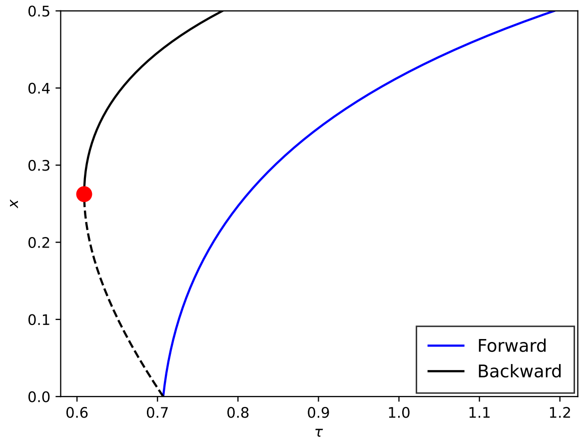


Figure 3: Bifurcation curves of system (1) in (τ, x) space for $\gamma = 1$. The values of β are $\beta = 1$ (blue) and $\beta = 2$ (black). The blue and black curves correspond respectively to the cases for which forward and backward type transcritical bifurcation occurs.

3 Generalization

Our goal now is to handle networks of arbitrary size using a similar approach to the methods showcased above. In order to force the lumped system to consist of exactly two equations, we must ensure that there are in a sense exactly two types of nodes. For each type, A and B, we determine the local structure of the hypergraph around a node of that type. For a node v of type $X \in \{A, B\}$, let $d_X(i)$ denote the number of order-two hyperedges containing the node v and i additional nodes of type X, where $i \in \{0, 1\}$. Similarly, let $r_X(i)$ denote the number of order-three hyperedges containing the node v and i additional nodes of type X, where $i \in \{0, 1, 2\}$. For a hypergraph structured as above, using $x(t)$ and $y(t)$ to denote, respectively, the probability that an arbitrary node of type A and B is infected at time t, the resulting ODE system is

$$\dot{x} = \tau\big[d_A(1)x + d_A(0)y\big](1-x) + \beta\big[r_A(2)x^2 + r_A(1)xy + r_A(0)y^2\big](1-x) - \gamma x, \quad (3a)$$

$$\dot{y} = \tau\big[d_B(1)y + d_B(0)x\big](1-y) + \beta\big[r_B(2)y^2 + r_B(1)xy + r_B(0)x^2\big](1-y) - \gamma y. \quad (3b)$$

Note that there does not exist a corresponding hypergraph for every combination of the structural parameters, certain relationships between them must hold. These are

$$n_A d_A(0) = n_B d_B(0), \quad n_A r_A(1) = 2n_B r_B(0), \quad n_B r_B(1) = 2n_A r_A(0),$$

where n_X denotes the total number of nodes of type X for $X \in \{A, B\}$. Keeping this in mind, the bifurcation analysis of the ODE system is conducted independently of the underlying hypergraph model for generality. This also means that we treat the structural parameters as continuous, instead of integer-valued. In this semester, I looked at a special case of system (3), where y appears linearly on the right-hand side, i.e. $r_A(0) = d_B(1) = r_B(2) = r_B(1) = 0$. I applied similar techniques as the ones used for the analysis of the two edge case on three nodes and found that the bifurcation landscape does not differ significantly.

References

[1] István Z. Kiss, Joel C. Miller, and Péter L. Simon. Mathematics of Epidemics on Networks: From Exact to Approximate Models. Interdisciplinary Applied Mathematics 46. Cham: Springer International Publishing, 2017. https://doi.org/10.1007/978-3-319-50806-1.

[2] István Z. Kiss, Joel C. Miller, Grzegorz A. Rempała, and Péter L. Simon. Mathematics of Epidemics on Networks: From Exact to Approximate Models. Unpublished draft of the second edition, July 2025

[3] Owen, L., Tuwankotta, J.M. Computation of fold and cusp bifurcation points in a system of ordinary differential equations using the Lagrange multiplier method. Int. J. Dynam. Control 10, 363–376 (2022). https://doi.org/10.1007/s40435-021-00821-4