Epidemics on Hypergraphs

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Individual Project II

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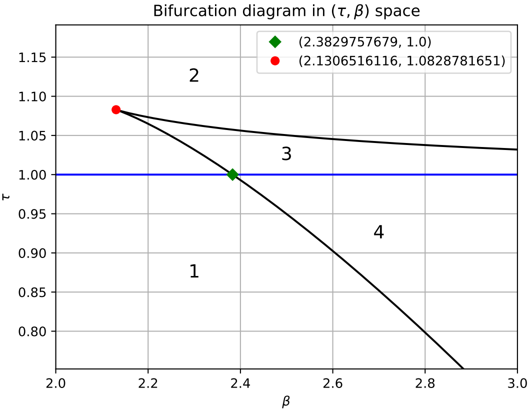
Recap

- SIS model (susceptible / infected)
- Hypergraph with hyperedges of order 2 and 3
- Rate parameters:
  - $\tau > 0$: infection across order 2 hyperedges
  - $\beta > 0$: infection across order 3 hyperedges (S: 1, I: 2)
  - $\gamma > 0$: recovery
- Bottom-up model
  - $\langle I_j \rangle(t) =$ probability that node j is infected at time t.
  - we derive ODEs
  - Goal: bifurcation diagram, steady states

N=3 nodes, with hyperedge of order 3, one edge



$$\dot{x} = \tau(1-x)x + \beta(1-x)xy - \gamma x,$$
$$\dot{y} = \beta(1-y)x^2 - \gamma y.$$

For steady states, we require

$$0 = x\left(\tau(1-x) + \beta(1-x)y - \gamma\right),$$
$$0 = \beta(1-y)x^2 - \gamma y.$$

We have: trivial/**disease-free** steady state + **endemic** steady states

Jacobian at $(0,0)$:

$$J(0,0) = \begin{pmatrix} \tau - \gamma & 0 \\ 0 & -\gamma \end{pmatrix}$$

Thus: stable: $\tau < \gamma$, unstable: $\tau > \gamma$, transcritical bifurcation at $\tau = \gamma$

For nonzero equilibria, we use a Lagrange-multiplier based method to compute fold bifurcations (nullclines meet tangentially):

Let $\gamma = 1$ and consider

$$F(x, y; \beta) = 0,$$
$$\nabla G(x, y; \beta) = \lambda \nabla F(x, y; \beta).$$

where $G(x, y, \beta) = \frac{1}{1-x} - \beta y$ and $F(x, y, \beta) = \beta(1-y)x^2 - y$.

This leads to the quartic equation in x (for fixed β):

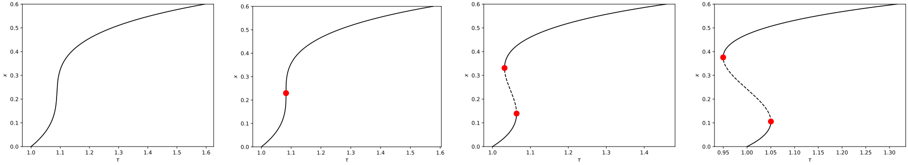
$$p(x) = \beta^2 x^4 - 2\beta^2 x^3 + (2\beta + 4\beta^2)x^2 - 2\beta^2 x + 1 = 0.$$

Since $p'' > 0$, we have at most 2 solutions.

At a cusp bifurcation, the two fold bifurcations coincide:

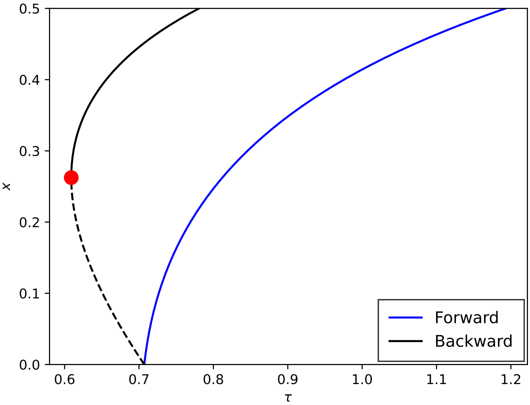
$$p(x) = \beta^2 x^4 - 2\beta^2 x^3 + (2\beta + 4\beta^2)x^2 - 2\beta^2 x + 1 = 0.$$
$$p'(x) = 4\beta^2 x^3 - 6\beta^2 x^2 + (4\beta + 8\beta^2)x - 2\beta^2 = 0$$

Numerical solution yields $\beta^* = 2.1306516116$.



(a) $\beta = 2.1$ (b) $\beta = \beta^*$ (c) $\beta = 2.3$ (d) $\beta = 2.5$

N=3 nodes, with hyperedge of order 3, two edges



$$\dot{x} = \tau(1-x)y + \beta(1-x)xy - \gamma x,$$
$$\dot{y} = 2\tau(1-y)x + \beta(1-y)x^2 - \gamma y.$$

For a steady state, we require

$$0 = \tau(1-x)y + \beta(1-x)xy - \gamma x,$$
$$0 = 2\tau(1-y)x + \beta(1-y)x^2 - \gamma y.$$

Jacobian at $(0,0)$:

$$J(0,0) = \begin{pmatrix} -\gamma & \tau \\ 2\tau & -\gamma \end{pmatrix}$$

Thus: stable: $\tau < \sqrt{2}\gamma$, unstable: $\tau > \sqrt{2}\gamma$, transcritical bifurcation at $\tau = \sqrt{2}\gamma$

Both equations are linear in y, so:

$$y = \frac{\gamma x}{(1-x)(\tau+\beta x)} = \frac{2\tau x + \beta x^2}{2\tau x + \beta x^2 + \gamma},$$

We obtain a quadratic equation for τ in terms of x:

$$A\tau^2 + B\tau + C = 0,$$

where $A = 2(1-x)$, $B = x\left(3\beta(1-x) - 2\gamma\right)$ and $C = \beta^2 x^2 (1-x) - (\beta\gamma x^2 + \gamma^2)$.
Solving:

$$\tau_{\pm}(x) = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Since $\tau_{-}(x) < 0 \ \forall x \in (0,1)$, therefore $\tau = \tau_{+}$.

Using implicit differentiation:

$$\tau'(x) = -\frac{A'\tau^2(x) + B'\tau(x) + C'}{2A\tau(x) + B},$$

where $A' = -2$, $B' = 3\beta(1-2x) - 2\gamma$ and $C' = 2\beta^2 x(1-x) - \beta^2 x^2 - 2\beta\gamma x$.

Solving

$$\tau'(0) = 0,$$

forward type transcritical bifurcation: $\beta < \frac{\sqrt{2}+2}{3}\gamma$, backward type transcritical bifurcation: $\beta > \frac{\sqrt{2}+2}{3}\gamma$.

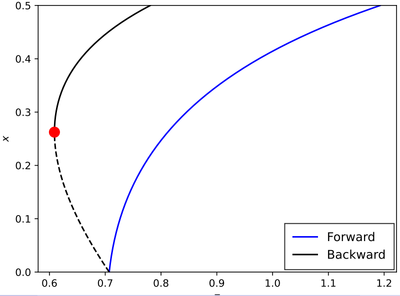
For fold bifurcations:

$$A'\tau^2(x) + B'\tau(x) + C' = 0$$

Multiplying by $(1-x)$ and adding $A\tau^2(x) + B\tau(x) + C = 0$, the τ^2 term is eliminated and we arrive at

$$\begin{aligned}\tau(x) &= -\frac{(1-x)C' + C}{(1-x)B' + B} \\ &= -\frac{2\beta^2 x(1-x)^2 - 2\beta\gamma x + \beta\gamma x^2 - \gamma^2}{3\beta(1-x)^2 - 2\gamma}.\end{aligned}$$

We solve this for x numerically and substitute back into $\tau(x)$.



Generalization

Goal: similar model with 2 equations for hypergraph of arbitrary size

Structural parameters:
For a node v of type $X \in \{A, B\}$, let

- $d_X(i)$: number of order-two hyperedges containing the node v and i additional nodes of type X, where $i \in \{0, 1\}$
- $r_X(i)$: number of order-three hyperedges containing the node v and i additional nodes of type X, where $i \in \{0, 1, 2\}$

$$n_A d_A(0) = n_B d_B(0), \quad n_A r_A(1) = 2 n_B r_B(0), \quad n_B r_B(1) = 2 n_A r_A(0)$$

$$
\begin{aligned}
\dot{x} &= \tau\left[d_A(1)x + d_A(0)y\right](1-x) \\
&\quad + \beta\left[r_A(2)x^2 + r_A(1)xy + r_A(0)y^2\right](1-x) - \gamma x, \\
\dot{y} &= \tau\left[d_B(1)y + d_B(0)x\right](1-y) \\
&\quad + \beta\left[r_B(2)y^2 + r_B(1)xy + r_B(0)x^2\right](1-y) - \gamma y.
\end{aligned}
$$

If $r_A(0) = d_B(1) = r_B(2) = r_B(1) = 0$, then y appears linearly and techniques used for the 2 edge case can be applied.

AI usage

I hereby declare that I used AI during my project work for the following:

- brainstorming, testing ideas
- assistance in writing Python codes