# DISSECTIONS OF ROOT POLYTOPES

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## INTRODUCTION

The Tutte polynomial of a graph can be determined in multiple ways. One of them is via embedding activity, another is via acyclic circuit signatures induced by weight functions on the edges, and there is a method using decision trees.

A dissection of a polytope is a set of pairwise disjoint simplices such that their union is the dissected polytope itself. The root polytope $Q_D$ associated with a directed graph $D = (V, E)$ is a special type of graph polytope that can be dissected in multiple ways as well, which is closely related to the Tutte polinomial. One construction is based on Jaeger trees, which are connected to embedding activity, while another arises from weight functions on the edge set. This motivates the present project: to investigate whether dissections can also be obtained using the decision-tree method or not.

## 1. ACYCLIC CIRCUIT SIGNATURES

The method using decision trees is highly connected to the one using acyclic circuit signatures, so first we have to understand how that method works.

We can associate a *root polytope* $Q_D$ to any finite directed graph $D = (V, E)$, defined as the convex hull

$$Q_D = \mathrm{Conv}\{\, \mathbf{x}_e \mid e \in E \,\} \subset \mathbb{R}^V,$$

where $\mathbf{x}_e$ denotes the vector with coordinate 1 for the head of $e$, coordinate $-1$ for the tail of $e$, and all other coordinates are 0. The *extended root polytope* is

$$\tilde{Q}_D = \mathrm{Conv}(\{\mathbf{0}\} \cup \{\, \mathbf{x}_e \mid e \in E \,\}) \subset \mathbb{R}^V.$$

Now let $D$ be a digraph, and let $C$ be a cycle in $D$. A *signed cycle* is an ordered partition $\overrightarrow{C} = C^+ \sqcup C^-$ so that $C^+$ contains the edges of $C$ going in one of the cyclic directions, and $C^-$ contains the edges of $C$ going in the other cyclic direction. Naturally, each cycle supports two signed cycles that can be obtained from one another by switching the roles of $C^+$ and $C^-$. We call $C^+$ and $C^-$ the two *arcs* of $\overrightarrow{C}$. The *vector* of a signed cycle $\overrightarrow{C}$, denoted by $\chi_{\overrightarrow{C}} \in \mathbb{Z}^E$, has coordinate $+1$ corresponding to $e$ if $e \in C^+$, has $-1$ if $e \in C^-$, and 0 if $e \notin C$.

A *circuit signature* $\sigma$ is a collection of signed cycles such that for each cycle $C$, exactly one of the signed cycles supported on $C$ is contained in $\sigma$. The signed cycle in $\sigma$ supported on $C$

is denoted by $\sigma(C)$. A circuit signature $\sigma$ is called *acyclic* if for any non-empty set of cycles $C_1,\dots,C_s$ and positive coefficients $a_1,\dots,a_s$, we have $\sum_{i=1}^s a_i\cdot\chi_{\sigma(C_i)}\neq\mathbf{0}$.

Let $w\colon E\to\mathbb{R}$ be a function, which in this context we will call a *weight function*. We say that $w$ is *generic* if we have $\sum_{f\in C^+}w(f)\neq\sum_{f\in C^-}w(f)$ for each signed cycle of $D$. This gives rise to a circuit signature in the following way.

**Definition 1.1.** For a generic weight function $w\colon E\to\mathbb{R}$, let the *induced circuit signature* $\mathrm{cir}^w$ be the one consisting of those signed cycles $\overrightarrow{C}=C^+\sqcup C^-$ that satisfy $\sum_{f\in C^+}w(f)>\sum_{f\in C^-}w(f)$.

It is easy to see that $\mathrm{cir}^w$ is acyclic, since the vector $\sum_{i=1}^s a_i\chi_{\sigma(C_i)}$, with each $a_i>0$, always has a positive scalar product with $w\in\mathbb{R}^E$.

**Proposition 1.2.** *A signature $\sigma$ is acyclic if and only if $\sigma=\mathrm{cir}^w$ for some generic weight function $w$.*

## 2. Dissections via acyclic circuit signatures

As mentioned earlier, acyclic circuit signatures can be used to construct dissections of extended root polytopes. We will denote the set of spanning forests of a digraph $D$ by $\mathrm{Base}(D)$.

**Definition 2.1.** We say that the spanning forest $F$ is *compatible* with the circuit signature $\sigma$ if $e\in\sigma(C(F,e))^+$ for each edge $e\in E-F$.

We denote the set of spanning forests of $D$ compatible with $\sigma$ by $\mathrm{Base}(D,\sigma)$.

**Definition 2.2.** A circuit signature $\sigma$ *long arc positive*, if for each cycle $C$ we have $|\sigma(C)^+|\geq|\sigma(C)^-|$.

**Proposition 2.3.** *Let $D$ be a directed graph. If $\sigma$ is an acyclic, long arc positive circuit signature, then* $\mathrm{Base}(D,\sigma)$ *is a dissecting forest set of $D$.*

*Proof.* We first show that for $F_1,F_2\in\mathrm{Base}(D,\sigma)$, the simplices $\tilde{Q}_{F_1}$ and $\tilde{Q}_{F_2}$ are interior disjoint.

Suppose for a contradiction that there is a point $\mathbf{p}\in\mathrm{int}(\tilde{Q}_{F_1})\cap\mathrm{int}(\tilde{Q}_{F_2})$. Then, $\mathbf{p}=\sum_{e\in F_1}\lambda_e\mathbf{x}_e=\sum_{e\in F_2}\mu_e\mathbf{x}_e$ with $\sum_{e\in F_1}\lambda_e<1$ and $\sum_{e\in F_2}\mu_e<1$ (the sums are smaller than 1 because $\mathbf{0}$ is a vertex in both simplices, and it has to have a positive coefficient), moreover $\lambda_e>0$ for each $e\in F_1$ and $\mu_e>0$ for each $e\in F_2$. Define $\lambda_e=0$ for $e\notin F_1$ and $\mu_e=0$ for $e\notin F_2$. As $F_1\neq F_2$, there exists some $e\in E$ with $\lambda_e\neq\mu_e$, whence we have a nontrivial linear relation $\mathbf{0}=\sum_{e\in E}(\lambda_e-\mu_e)\mathbf{x}_e$.

In this case there exists a signed circuit $\overrightarrow{C}$ such that $C^+\subseteq\{e\in E\mid\lambda_e-\mu_e>0\}$ and $C^-\subseteq\{e\in E\mid\lambda_e-\mu_e<0\}$. (This is proved in [3, Claim 2.9].) In particular, $C\subseteq F_1\cup F_2$. Also, for $e\in F_1-F_2$ we have $\lambda_e-\mu_e=\lambda_e>0$, in other words $(F_1-F_2)\cap C\subseteq C^+$, and for $e\in F_2-F_1$ we have $\lambda_e-\mu_e=-\mu_e<0$, whence $(F_2-F_1)\cap C\subseteq C^-$.

On the other hand, we can write $\chi_{\overrightarrow{C}}$ as a sum of vectors of signed fundamental cycles of $F_1$, as well as of $F_2$. Indeed $-\chi_{\overrightarrow{C}} = \sum_{e\in C-F_1} \chi_{\overrightarrow{C}(F_1,e)}$, which is true since each $e \in C - F_1 \subset (F_2 - F_1) \cap C$ is in $C^-$. Similarly $\chi_{\overrightarrow{C}} = \sum_{e\in C-F_2} \chi_{\overrightarrow{C}(F_2,e)}$, which is true since each $e \in C - F_2 \subset (F_1 - F_2) \cap C$ is in $C^+$. As $F_1, F_2 \in \mathrm{Base}(D,\sigma)$, the signed fundamental cycles $\overrightarrow{C}(F_i,e)$ for $i = 1,2$ and $e \notin F_i$, are in $\sigma$. Thus $\mathbf{0} = \sum_{e\in C-F_1} \chi_{\overrightarrow{C}(F_1,e)} + \sum_{e\in C-F_2} \chi_{\overrightarrow{C}(F_2,e)}$ is a positive linear combination of signed circuits in $\sigma$, contradicting the assumption that $\sigma$ is acyclic.

Now it is enough to show that $\tilde{Q}_D \subseteq \bigcup\{\tilde{Q}_F \mid F \in \mathrm{Base}(D,\sigma)\}$.

Let $\mathbf{p}$ be an arbitrary point in $\tilde{Q}_D$. By Caratheodory's theorem, we can choose $|V| - c(D) + 1$ affine independent generators such that $\mathbf{p}$ is in their convex hull. If $\mathbf{0}$ is one of these vectors, that means that we have $\mathbf{p} \in \tilde{Q}_F$ for some spanning forest $F$ (not necessarily in $\mathrm{Base}(D,\sigma)$).

Else if $\mathbf{0}$ is not one of the vectors, then we have $\mathbf{p} \in Q_S$ for some $S$ where $S \subset D$ contains a cycle $C$. Let us express this as $\mathbf{p} = \sum_{e\in S} \lambda_e \mathbf{x}_e$, where each $\lambda_e \geq 0$ and $\sum_{e\in S} \lambda_e = 1$. This implies

$$\mathbf{p} = \sum_{e\in S-C} \lambda_e \mathbf{x}_e + \sum_{e\in\sigma(C)^+} (\lambda_e - \varepsilon)\mathbf{x}_e + \sum_{e\in\sigma(C)^-} (\lambda_e + \varepsilon)\mathbf{x}_e,$$

where $\varepsilon = \min\{\lambda_e \mid e \in \sigma(C)^+\}$. The right hand side is a nonnegative linear combination, in which the coefficient of one edge of $C$ became 0, and where the sum of the coefficients is $1 - (|\sigma(C)^+| - |\sigma(C)^-|)\varepsilon \leq 1$ by the long arc positivity assumption. Therefore we can add $\mathbf{0}$ with a nonnegative coefficient to obtain a convex combination. We have thus found, just like in the previous case, a spanning forest $F \subset S$ such that $\mathbf{p} \in \tilde{Q}_F$.

Now we have to deal with the possibility that $F \notin \mathrm{Base}(D,\sigma)$. That means that for some $e_0 \notin F$ we have $e_0 \in \sigma(C(F,e_0))^-$. Let again $\mathbf{p} = \sum_{e\in F} \lambda_e \mathbf{x}_e + \lambda_0 \cdot \mathbf{0}$. We employ the same trick as before, that is, let $\lambda_{e_0} = 0$, and re-write our convex combination for $\mathbf{p}$ as

$$\mathbf{p} = \sum_{e\in F-C(F,e_0)} \lambda_e \mathbf{x}_e + \sum_{e\in\sigma(C(F,e_0))^+} (\lambda_e - \varepsilon)\mathbf{x}_e + \sum_{e\in\sigma(C(F,e_0))^-} (\lambda_e + \varepsilon)\mathbf{x}_e +$$

$$+ \big(\lambda_0 + (|\sigma(C(F,e_0))^+| - |\sigma(C(F,e_0))^-|)\varepsilon\big) \cdot \mathbf{0},$$

where $\varepsilon = \min\{\lambda_e \mid e \in \sigma(C(F,e_0))^+\}$. Since $\sigma$ is long arc positive, we have $|\sigma(C(F,e_0))^+| - |\sigma(C(F,e_0))^-| \geq 0$; furthermore, as $e_0 \in \sigma(C(F,e_0))^-$ by assumption, the coefficient $\lambda_{e_0} = 0$ has increased to $\varepsilon$ (or stayed the same, if $\varepsilon = 0$). Therefore the new expression is again a convex combination. As the coefficient of some $g \in \sigma(C(F,e_0))^+$ is 0 in the new convex combination, we may take $F' = F - g + e_0$ as another spanning forest of $D$ so that the associated simplex contains $\mathbf{p}$. Now let us show that in a well-defined sense, we have improved our situation.

Since $\sigma$ is acyclic, by Proposition 1.2., there exists a weight function $w\colon E \to \mathbb{R}$ such that $\sigma = \mathrm{cir}^w$. Let us consider the number $\mathrm{value}(F,\mathbf{p}) = \sum_{e\in E} \lambda_e w(e)$ associated to any spanning forest $F$ of $D$, where the $\lambda_e$ are the barycentric coordinates of $\mathbf{p}$ with respect to $\tilde{Q}_F$ (except for the one corresponding to $\mathbf{0}$), extended as 0 to the non-edges of $F$. Then for the two forests $F$

and $F'$ above, we have

$$\text{value}(F',\mathbf{p}) = \text{value}(F,\mathbf{p}) - \varepsilon \cdot \left( \sum_{e \in \sigma(C(F,e_0))^+} w(e) - \sum_{e \in \sigma(C(F,e_0))^-} w(e) \right) < \text{value}(F,\mathbf{p}),$$

where we used the the definition of $\text{cir}^w = \sigma$. This means that if there is any $e_0 \notin F$ such that $e_0 \in \sigma(C(F,e_0))^-$, then we can find another forest $F'$ such that $\mathbf{p} \in \tilde{Q}_{F'}$ and $\text{value}(F',\mathbf{p}) < \text{value}(F,\mathbf{p})$. As there are finitely many spanning forests, we cannot continue this indefinitely, which means that there is a forest $F$ such that $F \in \text{Base}(D,\sigma)$ and $\mathbf{p} \in \tilde{Q}_F$. $\square$

## 3. Weighting induced by decision trees

Consider a graph $G$. A *decision tree* is a perfect binary tree (every node has 0 or 2 children and all leaves are on the same level) $\Delta$ with a labelling $V(\Delta) \to E(G)$ such that along every path starting from the root and ending on a leaf, the labels of the nodes form a permutation of $E(G)$. In particular, the depth of every leaf is $|E(G)|$.

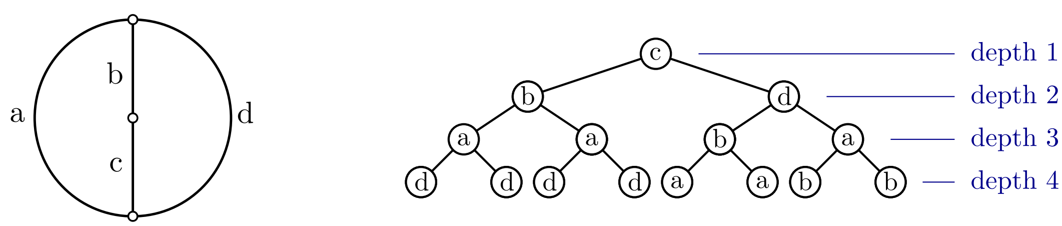


FIGURE 1. Example of a graph and an associated decision tree.

We modify our weight function by no longer using a uniform weighting scheme. Instead, we use weight functions induced by a decision tree. First, choose which edge will receive the largest weight and fix its value. Then partition the set of spanning trees into two classes according to whether the chosen heaviest edge is contained in the tree or not.

Among the trees containing this edge, we again choose an edge that will receive the second largest weight. The set of trees is then further partitioned according to whether this edge is contained in the given tree or not. We do the same in the other class of trees (those not containing the heaviest edge), so we similarly choose which edge will receive the second largest weight – possibly a different one from the first class of trees – fix its weight, and again partition the trees according to whether this edge belongs to them.

We continue this process iteratively until even the lightest edge has been assigned a weight in every spanning tree.

With this weighting scheme, the same edge may receive different weights in different spanning trees. However, the weights of all edges coincide for two trees up to the first edge (in the weight hierarchy) that belongs to exactly one of them.

We now call a spanning forest $F$ *compatible* with a given decision tree if for every edge $e \in E - F$ we have $e \in \sigma(C(F,e))^+$ with respect to its own weight function, that is, for $\sigma = \mathrm{cir}^{w_F}$, where $w_F$ denotes the weight function associated with $F$.

The question is whether the compatible trees form a dissecting forest set of $D$.

If we use only powers of 2 as weights, then the positive orientation of a cycle is determined by its largest-weight edge, since its weight exceeds the sum of all smaller weights. Consequently, in any cycle $C(F,e)$ of a compatible spanning tree, $e$ has the same cyclic orientation as the largest-weight edge of the cycle.

The first part of the proof of Proposition 2.3. also works in this setting. However, the second part falls, because as soon as we pass to a different spanning forest, the associated weighting changes as well. Consequently, the quantity $\mathrm{value}(F, \mathbf{p})$ is no longer guaranteed to decrease along the process.

This shows us the main difficulty of the decision-tree-induced weighting scheme, where the corresponding values are not directly comparable. Nevertheless, this structure still suggests a strong hierarchical behavior, which indicates that decision trees may still encode a meaningful combinatorial structure behind dissections of root polytopes, even though the original proof technique can no longer be applied directly.

The remaining challenge is therefore to determine whether an alternative monotonicity argument, or a different combinatorial invariant, can replace the role of the global weight function and lead to a proof that the compatible spanning forests indeed form a dissecting forest set.

## References

[1] T. Kálmán and L. Tóthmérész, *Graph minors, Ehrhart theory, and a monotonicity property*, arXiv:2409.18902 (2024).

[2] J. Courtiel, *A general notion of activity for the Tutte polynomial*, arXiv:1412.2081 (2014).

[3] L. Tóthmérész, *A geometric proof for the root-independence of the greedoid polynomial of Eulerian branching greedoids*, Journal of Combinatorial Theory, Series A **206** (2024), 105891.