# Dissections of root polytopes

## Individual project 2

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# Motivation

- The Tutte polynomial of a graph can be determined in multiple ways
  - Jaeger trees (embedding activity)
  - Acyclic circuit signatures
  - Decision trees
- The first two can be used to create dissections of root polytopes as well

Goal: Investigate whether dissections of root polytopes can also be obtained using the decision-tree method.

# Root polytopes

## Definition

Let $D = (V, E)$ be a finite directed graph. The *root polytope* of $D$ is

$$Q_D = \mathrm{Conv}\{\mathrm{x}_e \mid e \in E\} \subset \mathbb{R}^V,$$

where $\mathrm{x}_e$ has coordinate 1 at the head of $e$, coordinate $-1$ at the tail of $e$, and 0 elsewhere.

## Definition

The *extended root polytope* is

$$\tilde{Q}_D = \mathrm{Conv}(\{0\} \cup \{\mathrm{x}_e \mid e \in E\}) \subset \mathbb{R}^V.$$

# Circuit signatures

Let $D$ be a digraph, and let $C$ be a cycle in $D$.

**Definition**

A *signed cycle* is an ordered partition $\overrightarrow{C} = C^+ \sqcup C^-$ so that $C^+$ contains the edges of $C$ going in one of the cyclic directions, and $C^-$ contains the edges of $C$ going in the other cyclic direction.

- A *circuit signature* $\sigma$ is a collection of signed cycles such that for each cycle $C$, exactly one of the signed cycles supported on $C$ is contained in $\sigma$.
- A circuit signature $\sigma$ is called *acyclic* if no positive linear combination of its signed cycle vectors equals 0.

# Acyclic signatures from weight functions

Let $w : E \to \mathbb{R}$ be a generic weight function, which means $\sum_{f \in C^+} w(f) \neq \sum_{f \in C^-} w(f)$ for each signed cycle of $D$.

## Definition

The induced circuit signature $\mathrm{cir}^w$ consists of those signed cycles $\overrightarrow{C} = C^+ \sqcup C^-$ that satisfy $\sum_{f \in C^+} w(f) > \sum_{f \in C^-} w(f)$.

## Proposition

*A signature $\sigma$ is acyclic if and only if $\sigma = \mathrm{cir}^w$ for some generic weight function $w$.*

# Compatible spanning forests

Let $\mathrm{Base}(D)$ denote the spanning forests of $D$.

**Definition**

We say that the spanning forest $F$ is *compatible* with the circuit signature $\sigma$ if $e \in \sigma(C(F,e))^+$ for each edge $e \in E - F$.
The set of spanning forests of $D$ compatible with $\sigma$ is denoted by $\mathrm{Base}(D,\sigma)$.

**Definition**

A circuit signature $\sigma$ *long arc positive*, if for each cycle $C$ we have $|\sigma(C)^+| \geq |\sigma(C)^-|$.

Each forest determines a simplex of the extended root polytope.

# Dissections via acyclic circuit signatures

**Theorem**

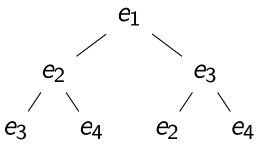
*Let $D$ be a directed graph. If $\sigma$ is an acyclic, long arc positive circuit signature, then $\mathrm{Base}(D, \sigma)$ is a dissecting forest set of $D$.*

Proof idea:

- Compatible simplices are internally disjoint.
- Every point of $\widetilde{Q}_D$ is covered by a compatible simplex.
  - If the forest is not compatible, add a violating edge to it, and discard another from the fundamental cycle, such that it still contains the point.
  - A value function strictly decreases during the process.

# Decision-tree-induced weighting

A *decision tree* is a perfect binary tree whose labels along each root-to-leaf path form a permutation of the edges.




Weights are assigned recursively according to the pathes:

- Choose the heaviest edge and fix its value.
- Partition the set of spanning forests into two classes according to whether the chosen heaviest edge is contained in the tree or not.

# Compatible forests for decision trees

The weighting depends on the spanning forest itself (the same edge may receive different weights in different spanning trees).

**Definition**

We now call a spanning forest $F$ *compatible* with a given decision tree if for every edge $e \in E - F$ we have $e \in \sigma(C(F,e))^+$ with respect to its own weight function, that is, for $\sigma = \mathrm{cir}^{w_F}$, where $w_F$ denotes the weight function associated with $F$.

Question:

Do these compatible forests form a dissecting forest set?

# Main difficulty

If powers of 2 are used as weights:

- The orientation of a cycle is determined by its heaviest edge.
- The first part of the previous proof still works.

Problem:

When we move from one spanning forest to another, then the edge weights change, and therefore the value function changes its meaning. $\Rightarrow$ The monotonicity argument breaks down.

Possible new approaches:

- Find a new monotonicity argument
- Construct a different combinatorial invariant

*Thank you for your attention!*

# AI usage

- Support in formulating and structuring presentation slides
- Assistance with LaTeX syntax
- Review of grammar and language accuracy