# Time Expanded Flows

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May 2026

## 1 Introduction

In this section, we define the classical flow problem and its time-expanded generalizations.

**Problem 1 (Classic flow problem)**
Let $G = (V, E)$ be a directed graph with source vertex $s$ and sink vertex $d$. Each edge $e \in E$ is associated with a $c_e \geq 0$ capacity. The goal is to determine flow values $f_e$ for each edge such that the following constraints are satisfied:

$$\max\Big(\sum_{ud\in E} f_{ud} - \sum_{dv\in E} f_{dv}\Big) \tag{1}$$

$$0 \leq f_e \leq c_e \qquad \forall e \in E \tag{2}$$

$$\sum_{uv\in E} f_{uv} - \sum_{vw\in E} f_{vw} = 0 \qquad \forall v \in V \setminus (s \cup d) \tag{3}$$

We can now define the time-expanded versions of the classical flow problem.

**Problem 2 (Time dependent discrete flow problem)**
Let $G = (V, E)$ be a directed graph with source vertex $s$ and sink vertex $d$, and an integer time limit $T \geq 0$. Each edge $e \in E$ is associated with a $C_e \colon \{0, 1, \ldots, T\} \to \mathbb{R}_{\geq 0}$ capacity function. The goal is find a $F_e \colon \{0, 1, \ldots, T\} \to \mathbb{R}_{\geq 0}$ function for each edge such that the following constraints are satisfied:

$$\max\Big(\sum_{ud\in E}\sum_{x=0}^{T} F_{ud}(x) - \sum_{dv\in E}\sum_{x=0}^{T} F_{dv}(x)\Big) \tag{4}$$

$$0 \leq F_e(\tau) \leq C_e(\tau) \qquad \forall e \in E, \tau \colon \{0, 1 \ldots T\} \tag{5}$$

$$\sum_{wv\in E}\sum_{x=0}^{\tau} F_{wv}(x) - \sum_{vu\in E}\sum_{x=0}^{\tau} F_{vu}(x) \geq 0 \qquad \begin{cases} \forall v \in V \setminus \{s, d\} \\ \forall \tau \in \{0, 1, \ldots, T\} \end{cases} \tag{6}$$

And similarly we can define the continuous version of the problem.

**Problem 3 (Time dependent continuous flow problem)**
Let $G = (V, E)$ be a directed graph with source vertex $s$ and sink vertex $d$, and a time limit $T \geq 0$. Each edge $e \in E$ is associated with a $C_e : [0, T] \to \mathbb{R}_{\geq 0}$ capacity function. The goal is find the a $F_e : [0, T] \to \mathbb{R}_{\geq 0}$ integrable function for each edge such that the following constraints are satisfied:

$$\max(\sum_{ud \in E} \int_0^T F_{ud}(x)dx - \sum_{dv \in E} \int_0^T F_{dv}(x)dx) \tag{7}$$

$$0 \leq F_e(\tau) \leq C_e(\tau) \qquad \forall e \in E, 0 \leq \tau \leq T \tag{8}$$

$$\sum_{wv \in E} \int_0^\tau F_{wv}(x)dx - \sum_{vu \in E} \int_0^\tau F_{vu}(x)dx \geq 0 \begin{cases} \forall v \in V \setminus \{s, d\} \\ \forall 0 \leq \tau \leq T \end{cases} \tag{9}$$

It can be seen that all three problems are described by three formulas. The first formula determines the value of the objective function, namely how much more flow enters the vertex $d$ than leaves it. The second constraint gives a limit on the maximum flow value on a given edge at a given time. The third formula represents flow conservation in the classical flow problem; this can be generalized in the other two models such that, up to a given time, the amount of incoming flow to a vertex is at least as large as the amount of outgoing flow from the vertex.

In the time-dependent discrete and continuous versions, capacity functions are used instead of capacities on the edges (which are defined only on integer values or on the entire interval $[0, T]$), and accordingly their values may vary over time. Similarly, the solution $F$ may also vary over time. Because of this, instead of the equalities (1) and (3), summation and integration are required to compute the corresponding quantities.

In the continuous version, we try to find integrable functions $F$, since otherwise it could not be evaluated.

# 2 Time dependent discrete problem

In this chapter, we will deal with the time-dependent discrete flow problem.

We define the maximum dynamic flow and the minimum dynamic cut in a more general model. We examine how the time-dependent discrete flow problem can be reduced to the classical flow problem (possibly with a significant increase in the number of vertices), with the help of which we can prove the maximum flow – minimum cut theorem in the time-dependent case.

## 2.1 Definitions

Let us consider the general case of the previously defined time-dependent discrete flow, where traversal times, time-varying capacity constraints, and storage limits are also present.

**Problem 4 (General time dependent discrete flow problem)**
Let $G = (V, E)$ be a directed graph with source vertex $s$ and sink vertex $d$, and an integer time limit $T \geq 0$. Each edge $e \in E$ is associated with a $C_e\colon \{0, 1, \dots, T\} \to \mathbb{R}_{\geq 0}$ capacity function, and a $L_e\colon \{0, 1, \dots, T\} \to \mathbb{R}_{\geq 0}$ time dependent traversal time. Also for each $v \in V$ vertex there is a $C_e\colon \{0, 1, \dots, T\} \to \mathbb{R}_{\geq 0}$ storage function. The goal is find a $F_e\colon \{0, 1, \dots, T\} \to \mathbb{R}_{\geq 0}$ function for each edge such that the following constraints are satisfied:

$$\max(\sum_{ud \in E} \sum_{x=0}^{T} F_{ud}(x) - \sum_{dv \in E} \sum_{x=0}^{T} F_{dv}(x)) \tag{10}$$

$$0 \leq F_e(\tau) \leq C_e(\tau) \qquad \forall e \in E, \tau\colon \{0, 1 \dots T\} \tag{11}$$

$$0 \leq \sum_{wv \in E} \sum_{\substack{0 \leq x \leq \tau \\ x + L_{wv}(x) \leq \tau}} F_{wv}(x) - \sum_{vu \in E} \sum_{x=0}^{\tau} F_{vu}(x) \leq C_v(\tau) \begin{cases} \forall v \in V \setminus \{s, d\} \\ \forall \tau \in \{0, 1, \dots, T\} \end{cases} \tag{12}$$

The meaning of the (12) equation that the stored flow in $v$ vertex and $\tau$ time is at most the storage capacity. The two half of the sum is asymmetrical because of the traversing times.

**Definition 1.** Let us denote by $F^*$ the value of the time-dependent discrete flow.

**Definition 2.** $S$ cut: for each $v \in V$ vertex a $S_v \subseteq 0, 1, \dots T$ subset, $S_s = \{0, 1, \dots T\}$, $S_d = \emptyset$.

**Definition 3.** size of the cut $S$ :

$$cap(S) = \sum_{v \in V} \sum_{\substack{0 \leq x \leq T \\ x \in S_v, x+1 \notin S_v}} C_v(x) + \sum_{uv \in e} \sum_{\substack{0 \leq x \leq T - L_{uv} \\ x \in S_u, x + L_{uv} \notin S_v}} C_{uv}(x) \tag{13}$$

**Definition 4.** Minimal cut:

$$cap(S^*) = \min_{S \text{cut}} cap(S) \tag{14}$$

**Theorem 1**
In the time dependent discrete model $F^* = cap(S^*)$

For the proof, we reduce the time-dependent flow problem to the classical flow problem, for which we already know that the value of the maximum flow and the size of the minimum cut are equal.

## 2.2 Time expanded graph

For easier notation, we introduce a uniform storage limit of $\infty$ at the vertices $s$ and $d$.

We will construct the time expanded graph of the, which is denoted by $G^{TEG}$. $G^{TEG}$ has $|V| \cdot (T+1)$ vertices, each vertex is characterized by a pair $(v \in V, 0 \leq \tau \leq T)$.

For each edge $uv \in E$ and for each time instant $0 \leq \tau, \tau + L_{uv}(\tau) \leq T$, we add an edge of capacity $C_{uv}(\tau)$ between $(u, \tau)$ and $(v, \tau + L_{uv}(\tau))$.

In addition, for each vertex $v \in V$ and for each time $0 \leq \tau < T$, we add an edge of capacity $C_v(\tau)$ between $(v, \tau)$ and $(v, \tau + 1)$.

**Theorem 2**

The value of the maximum dynamic *sd* flow on $G$ is equal to the value of the maximum static $(s, 0) - (d, T)$ flow on $G^{TEG}$.

*Proof.* If we take a dynamic flow on $G$, then from this we can construct a static flow of the same value on $G^{TEG}$ by assigning, for each edge $uv$ and time instant $\tau$, the flow value $F_{uv}(\tau)$ to the edge between $(u, \tau)$ and $(v, \tau + L_{uv}(\tau))$ in $G^{TEG}$.

Afterwards, we only need to determine the flow values on the edges between $(v, \tau)$ and $(v, \tau + 1)$, which, due to equation (12), will satisfy the classical flow constraints. The correspondence in the other direction works essentially in the same way. $\square$

**Theorem 3**

The value of the minimum dynamic cut on $G$ is equal to the value of the minimum $(s, 0) - (d, T)$ cut on $G^{TEG}$.

*Proof.* In the classical flow, a cut can be defined by a set $S'$, where $(s, 0) \in S'$ and $(d, T) \notin S'$. The size of the cut is the sum of the edges going from $S'$ to $\overline{S'}$. In $G^{TEG}$, there are edges of infinite capacity between $(s, \tau)$ and $(s, \tau + 1)$, as well as between $(d, \tau)$ and $(d, \tau + 1)$. Because of this, $(s, \tau) \in S'$ and $(d, \tau) \notin S'$ holds for all $0 \leq \tau \leq T$.

Thus, in the dynamic flow, the cut $S$ and the classical flow cut $S'$ can be uniquely mapped to each other, and from the construction of $G^{TEG}$ it follows that their sizes are also equal. $\square$

From these two statements it follows that the time-dependent discrete flow problem can also be viewed as a static flow problem. In the classical case, it is a fundamental theorem that the value of the maximum flow and the minimum cut are equal, from which Theorem 2.1 immediately follows.

In $G^{TEG}$, a maximum feasible flow can be found using several algorithms in polynomial time in the number of vertices. The weakness of this approach appears when $|V|$ is small and $T$ is very large, and the functions can be described with few parameters, since in this case this approach is not polynomial in the size of the input.

Nevertheless, it is extremely useful; in many cases statements can be reduced to this setting, and the known theorems can be transferred to the time-dependent version.

# 3 Continuous Flow Problem

In this chapter, we will deal with the continuous model. We examine how the maximum flow can be approximated, using the method defined in the previous chapter based on the time expanded graph.

## 3.1 Real life applications

For the analysis of communication processes such as Starlink or similar systems, time-dependent flows are highly advantageous.

A given amount of information must be delivered, using drones or satellites, to a designated location as quickly as possible.

The trajectories of drones are predictable, and two drones can communicate efficiently only if they are sufficiently close to each other. Since the amount of information that can be transmitted between two drones varies over time, static flow is not sufficient, and time-dependent dynamic flows are needed to model the system. When drones are close enough, we may assume that the information transfer happens instantaneously.

In addition, each drone can store only a limited amount of information, so storage constraints can also be introduced; however, in practice this capacity is orders of magnitude larger than the amount of information to be transmitted, so in most of this chapter we assume that there are no storage constraints.

## 3.2 Problem

Let us recall the time-dependent flow problem introduced in the first chapter.

**Problem 5 (Time dependent continuous flow problem)**
Let $G = (V, E)$ be a directed graph with source vertex $s$ and sink vertex $d$, and a time limit $T \geq 0$. Each edge $e \in E$ is associated with a $C_e : [0, T] \to \mathbb{R}_{\geq 0}$ capacity function. The goal is find the a $F_e : [0, T] \to \mathbb{R}_{\geq 0}$ integrable function for each edge such that the following constraints are satisfied:

$$\max(\sum_{ud \in E} \int_0^T F_{ud}(x)dx - \sum_{dv \in E} \int_0^T F_{dv}(x)dx) \tag{15}$$

$$0 \leq F_e(\tau) \leq C_e(\tau) \qquad \forall e \in E, 0 \leq \tau \leq T \tag{16}$$

$$\sum_{wv \in E} \int_0^\tau F_{wv}(x)dx - \sum_{vu \in E} \int_0^\tau F_{vu}(x)dx \geq 0 \begin{cases} \forall v \in V \setminus \{s, d\} \\ \forall 0 \leq \tau \leq T \end{cases} \tag{17}$$

This is the special, simpler case; in the more general case, there is also a storage capacity $C_v : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ for the vertices $v \in V \setminus \{s, d\}$.

If we use the:

$$F_v(\tau) = \sum_{e_{wv} \in E} \int_0^\tau F_e(x)dx - \sum_{e_{vu} \in E} \int_0^\tau F_e(x)dx$$

notation, then instead of, (17) we get the following condition.

$$0 \le F_v(\tau) \le C_v(\tau) \qquad \forall v \in V \setminus \{s,d\}, \forall 0 \le \tau \le T \tag{18}$$

In the simpler case we will ignore $C_v(\tau)$, so we might assume that for each $v \in V$ and $0 \le \tau \le T$ $C_v(\tau) \equiv \infty$.

**Definition 5.** Let us denote by $F^*$ the value of the time-dependent discrete flow.

We may assume that the graph is simple, since if $e_1$ and $e_2$ are both edges between $u$ and $v$, then instead of these we can introduce a single edge $e$ with $C_e = C_{e_1} + C_{e_2}$.

For easier notation later on, we also assume that there are no edges $uv$ and $vu$ simultaneously in the graph.

Due to practical test cases, we require that for every $e \in E$, the function $C_e$ is integrable, and in particular piecewise continuous. We also require that the function $C_e$ is bounded. If $C_e$ changes its value only at finitely many points, the solution remains the same (since values at finitely many points do not affect integration), therefore we may assume that $C_e$ is right-continuous, and in the solutions we also look for flows where the functions $F_e$ are right-continuous.

The problem can be formulated in several ways.

In the problem, for a given value $T$, we search for the maximum flow. In practice, we typically consider a given value $F^*$ and search for the smallest value $T'$ such that the value of the maximum dynamic flow up to time $T'$ is at least $F^*$.

If we are satisfied with an arbitrarily precise approximation, then the two problems can easily be reduced to each other.

We observe that for larger values of $T$, the value of the maximum flow is also larger. Based on this, if we plot $F^*$ as a function of $T$, we obtain a monotonically increasing function, on which we can approximate the smallest value $T'$ for which the value of the maximum continuous flow is at least $F^*$ using binary search with arbitrary precision.

A similar reduction works in the other direction as well.

In this work, we consider the version defined in the problem, i.e., $T$ is fixed and we search for a flow corresponding to it.

## 3.3 Approximation algorithm

Similarly to the previous section we also try to reduce the continuous flow problem to the classical flow problem. In detail, we again consider the version without capacities.

For every $n \ge 1$, we define the time expanded graphs $G_{TEG}(n)$ and $G^{TEG}(n)$ such that the maximum flow in $G_{TEG}(n)$ is a lower bound, and in $G^{TEG}(n)$ an upper bound on the value of the continuous maximum flow, and in both $G_{TEG}(n)$ and $G^{TEG}(n)$ the value of the maximum static flow converges to $F^*$.

The idea will be very similar to computing the lower and upper Riemann sums of a function.

Here, $G_{TEG}(n)$ and $G^{TEG}(n)$ will be constructed with a similar logic as the time-dependent graph $G_{TEG}$ defined in the discrete case, but they represent different objects. Moreover, in the discrete case the maximum flow in the time-dependent graph gives an exact value, which will not be true in the continuous case.

In the simpler case of the continuous model, we try to estimate the value of the maximum flow. Let us choose a value $n \geq 1$, and for this construct the graphs $G_{TEG}(n)$ and $G^{TEG}(n)$. Let $\tau = \frac{T}{n}$.

In both $G_{TEG}(n)$ and $G^{TEG}(n)$, there are $|V| \cdot (n+1)$ vertices, and each vertex corresponds to a pair $(v, i)$ with $v \in V$ and $0 \leq i \leq n$.

For every pair $(v, i)$ with $i < n$, we add an edge of infinite capacity from $(v, i)$ to $(v, i+1)$. This corresponds to infinite storage capacity at the given vertex.

In addition, for each edge $e = uv$ and each time instant $0 \leq i < n$, we define

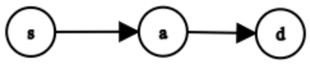
$$K_e(i) = \int_{i \cdot \tau}^{(i+1) \cdot \tau} C_e(x)\, dx$$

That is, on a given edge, this represents the maximum amount of flow that can pass during the interval between the two selected time points.

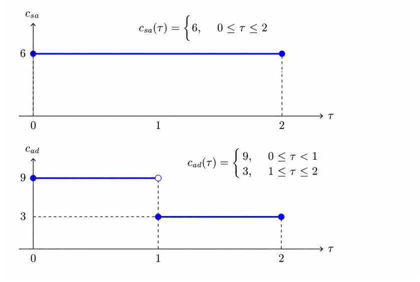
For every $e \in E$ and $0 \leq i < n$, in $G_{TEG}(n)$ we add an edge from $(u, i)$ to $(v, i+1)$, while in $G^{TEG}(n)$ we add an edge from $(u, i)$ to $(v, i)$, each with capacity $K_e(i)$.

Let $G_1(n)$ denote the value of the maximum static $(s, 0) - (d, n)$ flow in $G_{TEG}(n)$, and $G_2(n)$ the corresponding value in $G^{TEG}(n)$, while for the time-dependent flow we continue to use $F^*$.

Let us consider a simple example for the continuous problem, for the maximum flow, and for $G_{TEG}(n)$ and $G^{TEG}(n)$.

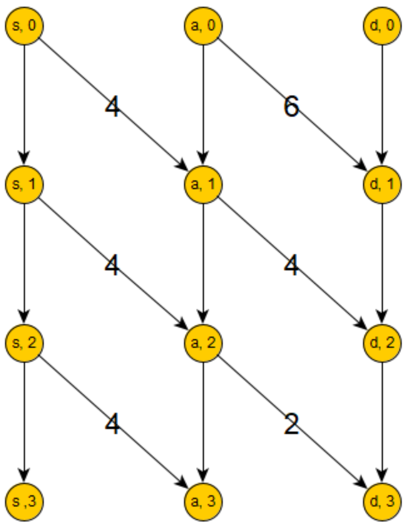


(a) The edges of the graph, $T = 2$

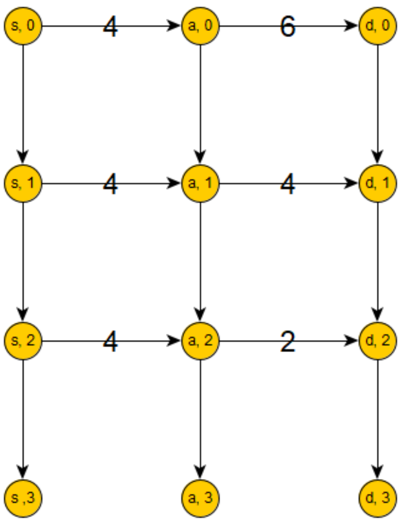


(b) The time depending capacity of the edges

$$\text{In the optimal solution: } F_{sa}(\tau) = F_{ad}(\tau) = \begin{cases} 6, & 0 \le \tau < 1 \\ 3, & 1 \le \tau \le 2 \end{cases}$$



(a) $G_{TEG}(3)$, edges without a number have $\infty$ capacity



(b) $G^{TEG}(3)$ edges without a number have $\infty$ capacity similarly

For $N = 3$ we can observe that $G_1(3) = 6 < F^* = 9 < G_2(3) = 10$

**Theorem 4**

$$G_1(n) \leq F^* \leq G_2(n)$$
$$G_1(n) \xrightarrow{n\to\infty} F^*$$
$$G_2(n) \xrightarrow{n\to\infty} F^*$$

The proof is quite technical; the first two inequalities are obvious after constructing and understanding the graphs, while the two convergences are not trivial, but it is worth computing them in order to easily determine the accuracy of the estimate for a given $n$.

*Proof.*

- $G_1(n) \leq F^*$

  First we construct the graph $G_{TEG}(n)$ and compute the maximum flow on it. For each $uv$ edge and $0 \leq \tau < n$ moment, let $f_e(\tau)$ the flow on the edge from $(u, \tau)$ to $(v, \tau + 1)$. We can choose the functions $F_e$ on the graph $G$ such that

  $$\int_{i\cdot\tau}^{(i+1)\cdot\tau} F_e(x)\,dx = f_e(i)$$

  holds for every edge $e$ and every time instant $0 < i < n$.

  It is easy to see that with such a choice, the functions $F_e$ provide a feasible solution, and the total flow coincides with $G_1(n)$.

- $G_2(n) \geq F^*$

  Let us take the optimal solution for the time-dependent graph $G$. Then, after constructing $G^{TEG}(n)$, for every edge $uv \in E$ and every time instant $0 \leq i < n$, we assign the flow value on the edge between $(u, i)$ and $(v, i+1)$ as

  $$\int_{i\cdot\tau}^{(i+1)\cdot\tau} F_e(x)\,dx.$$

  In addition, for every pair $v \in V$ and $0 \leq i < n$, we also need to determine the flow value on the edges between $(v, i)$ and $(v, i+1)$ in $G^{TEG}(n)$, which is always the excess generated at vertex $v$ at time $i$.

  In this way, we obtain a feasible solution, where the value of the flow is $F^*$, thus the statement is proved.

- $G_1(n) \xrightarrow{n\to\infty} F^*$

  Before the proof, let us consider two lemmas:

  **Lemma 1**
  In a graph $G$, the value of the maximum $sd$ static flow can increase by at most $x$ after adding an edge with capacity $x$.

  *Proof.* The value of the maximum flow is equal to the minimum cut, and if we extend the original cut with the added edge, we obtain a cut in the new graph whose value has increased only by $x$. □

**Lemma 2**
In a directed graph $G = (V, E)$, where each edge has an associated value $c_e$, and for fixed source and sink vertices $s$ and $d$, if we assign to each $e \in E$ a value $0 \leq f_e \leq c_e$, then the value of the maximum flow in $G$ is at least:

$$\sum_{ud \in E} f_{ud} - \sum_{dv \in E} f_{dv} - \sum_{v \in V \setminus (s,d)} \max(0, \sum_{vw \in E} f_{vw} - \sum_{uv \in E} f_{uv}) \tag{19}$$

That is, compared to the usual formula, we also subtract from each internal vertex the deficit (i.e., by how much the sum of outgoing edges exceeds the sum of incoming edges), if it is positive.

*Proof.* If none of the vertices are deficient, then we can reduce the edge values so that in every internal vertex the sum becomes exactly 0, without decreasing the sum of edges entering $d$. In this case, the last term in equation (2) becomes 0 for all $v \in V$, and we obtain the well-known formula of the classical flow problem.

In the case of deficient vertices, we apply the previous lemma multiple times by adding a corresponding edge of capacity to each deficient vertex $v$, and if we use this edge, then the total deficit decreases by the capacity of the edge. Repeating this step, we reach a state where there are no deficient vertices, for which the statement holds. $\square$

Let us consider the maximum dynamic flow in $G$. In $G_{TEG}(n)$, for an edge $e = uv$ and a value $i$, we denote the capacity of the edge between $(u, i)$ and $(v, i+1)$ by $Z_e(i)$, where

$$Z_e(i) := \int_{i \cdot \tau}^{(i+1) \cdot \tau} F_e(x)\, dx.$$

In the graph $G_{TEG}(n)$, we denote the flow value on the edges between $(v, i)$ and $(v, i+1)$ by $Z_v(i)$. We choose it such that we compute the sum of incoming edges into $(v, i)$ and subtract the sum of outgoing edges from $(v, i)$; let the difference be denoted by $X$. Then $Z_v(i) = \max(X, 0)$ will be the value of the edge. Here, for $i > 0$, the incoming edges already include the value $Z_v(i-1)$ as well.

Let us compute what lower bound can be given for $G_1(n)$ using the lemmas after choosing the $Z$ values.

The sum of edges entering the vertex $(d, n)$ is again $F^*$ (i.e., this is the value to which it must converge).

For every vertex $v \in V$, we compute the sum of deficits in $G_{TEG}(n)$ over the vertices $(v,0),(v,1),\ldots,(v,n)$. Let us focus on a fixed vertex $v$, and denote by $in(i)$ and $out(i)$ the sum of capacities of edges entering and leaving the vertex up to time $i$, respectively (i.e., we do not include the edges between $(v,j)$ and $(v,j+1)$).

$$in(i) = \sum_{j=0}^{i-1} \sum_{uv \in E} Z_u(j)$$

$$out(i) = \sum_{j=0}^{i} \sum_{vu \in E} Z_u(j)$$

We note that we are not summing over the same ranges, since in $G_{TEG}(n)$ the edges connect different layers.

Let $h(i)$ denote the deficit at the vertex $(v,i)$, and let $H(i)$ denote the sum of the values $h(i)$ from the 0th vertex up to the $i$th vertex. For simplicity, let $h(-1) = 0$.

$$H_i = \max(H(i-1), out(i) - in(i))$$

Since $h(i)$ cannot be negative, we have $H(i) \geq H(i-1)$, and also $H(i) \geq out(i) - in(i)$, since the total capacity of outgoing edges is at least this much (not including the edge between $(v,i)$ and $(v,i{+}1)$, which, if positive, would only increase the deficit). However, it can also be seen that $H(i)$ cannot be larger than both.

Based on this,

$$H(n) = \max_{0 \leq i < n} (out(i) - in(i))$$

We also know that $in(i+1) \geq out(i)$, since in the continuous flow problem this follows from equation (18).

Using this, we obtain

$$H(n) \leq \max_{0 \leq i < n} (in(i+1) - in(i)) \leq \sum_{uv \in E} \max_{0 \leq i < n} \int_{i \cdot \tau}^{(i+1) \cdot \tau} F_{uv}(x)\, dx$$

In the first inequality, we upper-bounded $out(i)$ by $in(i+1)$, and in the second one we upper-bounded $in(i+1) - in(i)$ by considering, for each edge entering $v$, on which subinterval of length $\tau$ the integral was maximal.

If we compute this for every internal vertex $v$, then we obtain the sum of the deficits.

We can further upper-bound this by summing these maxima over all edges, and using the condition $F_{uv}(x) \leq C_{uv}(x)$, we instead compute using the

$C$ values. This is important, since in this way the value can be determined without knowing the flow itself.

Putting these together:

$$F^* - F_1(n) \leq \sum_{e \in E} \max_{0 \leq i < n} \int_{i \cdot \tau}^{(i+1) \cdot \tau} C_e(x)\, dx \tag{20}$$

We assumed that the functions $C_e$ are bounded; let $P = \max C_e(x)$. From this it follows that

$$F^* - F_1(n) \leq |E| \cdot P \cdot \tau \tag{21}$$

From this, convergence follows immediately, and we can easily choose a suitable value of $n$ for a desired error bound.

- $F_2(n) \xrightarrow{n \to \infty} F^*$

  Instead, let us prove that $F_2(n) \xrightarrow{n \to \infty} = F_1(n) \xrightarrow{n \to \infty}$.

  The proof is very similar to the previous one. For an arbitrary $n$, we construct the graphs $G_{TEG}(n)$ and $G^{TEG}(n)$, solve the static flow problem on $G^{TEG}(n)$, write these capacities onto the edges of $G_{TEG}(n)$, and compute the sum of the missing capacities at the vertices.

  The computation is essentially the same, and the same result is obtained.

$$F_2(n) - F_1(n) \leq |E| \cdot P \cdot \tau \tag{22}$$

From this, the statement to be proven follows.

Summing up: $F_1(n) \leq F^* \leq F_2(n)$ and $F_2(n) - F_1(n) \leq |E| \cdot P \cdot \frac{T}{n}$, where $P = \max C_e(\tau)$. □

This is useful, since for any $\epsilon > 0$, for $n = \frac{|E| \cdot P \cdot T}{\epsilon}$ we have $F_2(n) - F_1(n) \leq \epsilon$. Based on this, if we have a subroutine for computing $\int_0^T c_e(x)\, dx$, we can give an algorithm with arbitrary precision by reducing the problem to the classical flow problem.

[5] [3] [4] [2] [1]

## References

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## Use of Artificial Intelligence

I declare that during the preparation I used AI for the following tasks:

| Task | Tools used |
|---|---|
| Latex formatting | GPT 5.5 |
| English translation | GPT 5.5 |
| Literature rewiew | GPT 5.5 |