Time Expanded Flows

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Classic Flow Problem

Let $G = (V, E)$ be a directed graph with source vertex s and sink vertex d. Each edge $e \in E$ is associated with a $c_e \geq 0$ capacity. The goal is to determine flow values f_e for each edge such that the following constraints are satisfied:

$$\max\Big(\sum_{ud \in E} f_{ud} - \sum_{dv \in E} f_{dv}\Big) \tag{1}$$

$$0 \leq f_e \leq c_e \qquad \forall e \in E \tag{2}$$

$$\sum_{uv \in E} f_{uv} - \sum_{vw \in E} f_{vw} = 0 \qquad \forall v \in V \setminus (s \cup d) \tag{3}$$

Main goal: generalize the classic flow problem for time expanded version

Time dependent discrete flow

- $T \geq 0$ time limit is given
- Each edge $e \in E$ is associated with a $C_e\colon \{0, 1, \ldots, T\} \to \mathbb{R}_{\geq 0}$ capacity function.

$$\max\Big(\sum_{ud \in E} \sum_{x=0}^{T} F_{ud}(x) - \sum_{dv \in E} \sum_{x=0}^{T} F_{dv}(x)\Big) \tag{4}$$

$$0 \leq F_e(\tau) \leq C_e(\tau) \qquad \forall e \in E, \tau\colon \{0, 1 \ldots T\} \tag{5}$$

$$\sum_{wv \in E} \sum_{x=0}^{\tau} F_{wv}(x) - \sum_{vu \in E} \sum_{x=0}^{\tau} F_{vu}(x) \geq 0 \quad \begin{cases} \forall v \in V \setminus \{s, d\} \\ \forall \tau \in \{0, 1, \ldots, T\} \end{cases} \tag{6}$$

LaTeX

Time expanded graph

Goal: reduce the discrete flow problem to the classic case We will construct a graph G_{TEG}

- Each node is represented by a $(v \in V, 0 \leq \tau \leq T)$
- $(v, \tau) \rightarrow (v, \tau + 1)$ edge, with ∞ capacity
- $(u, \tau) \rightarrow (v, \tau)$ edge, with $C_{uv}(\tau)$ capacity

Maximal time dependent flow in G is the same as maximal discrete flow in G_{TEG}

- We can prove the Minimum Cut Maximum Flow theorem in the time dependent version
- Also works in more general cases
- Polinomial algorithm, if T is small
- Not useful if $T >> |V|$, and the functions are simple

Continuous Flow problem

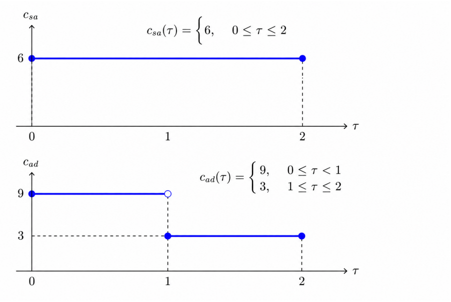
Each edge $e \in E$ is associated with a $C_e[0, T] \to \mathbb{R}_{\geq 0}$ capacity function.

$$\max\Big(\sum_{ud \in E} \int_0^T F_{ud}(x)dx - \sum_{dv \in E} \int_0^T F_{dv}(x)dx\Big) \tag{7}$$

$$0 \leq F_e(\tau) \leq C_e(\tau) \qquad \forall e \in E, 0 \leq \tau \leq T \tag{8}$$

$$\sum_{wv \in E} \int_0^\tau F_{wv}(x)dx - \sum_{vu \in E} \int_0^\tau F_{vu}(x)dx \geq 0 \begin{cases} \forall v \in V \setminus \{s, d\} \\ \forall 0 \leq \tau \leq T \end{cases} \tag{9}$$

LaTeX



The time dependent capacity of the edges

In the optimal solution:

$$F_{sa}(\tau) = F_{ad}(\tau) = \begin{cases} 6, & 0 \le \tau < 1 \\ 3, & 1 \le \tau \le 2 \end{cases}$$

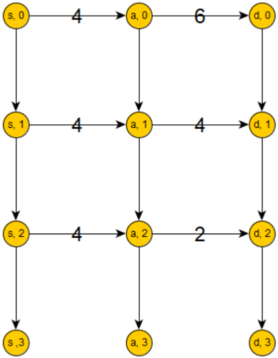


Figure: $G^{TEG}(3)$ edges without a number have ∞ capacity similarly

The max flow in G^{TEG} converges to the dynamic max flow.

Thank you for your attention!