Chromatic number of odd distance graphs on a circle and geodetic angle decomposition

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MOTIVATION

$\boldsymbol{G^{odd}}$:

- V(G): points of the Euclidean plane
- E(G): odd length segments

What is the chromatic number of the odd distance graph $\boldsymbol{G^{odd}}$?

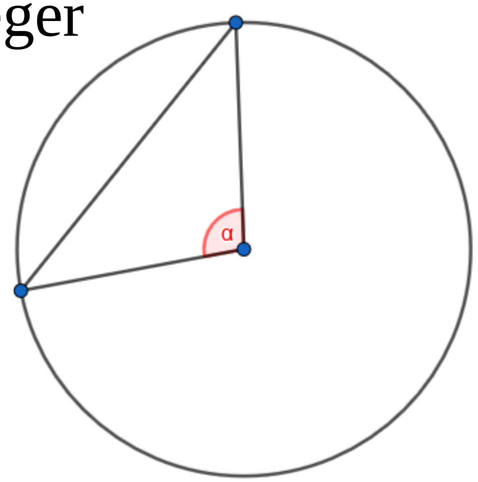
Davies: Can’t be colored with finite many colors→What about circles on the plane?

OVERVIEW

- G^r:
  - $V(G^r)$: points of a circle
  - $E(G^r)$: odd distances between vertices
- Main question: How many colors are needed to color G^r?
- Depends largely on the radius, only countable many cases are interesting
- $r = p\sqrt{Q}, \;\; p \in \mathbb{Q}$, Q square free integer

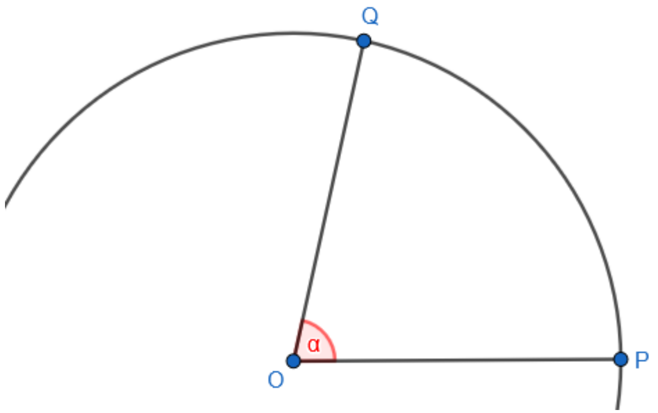
Residual of an Angle

- Let α be an angle, such that:
  - $\sin(\alpha) = p\sqrt{D}$, where p is rational, and D is a square-free integer
  - $\cos(\alpha)$ is rational
  - Then α has a residual D
- Most of the angles doesn’t have a residual
- Central angle of a chord has a residual

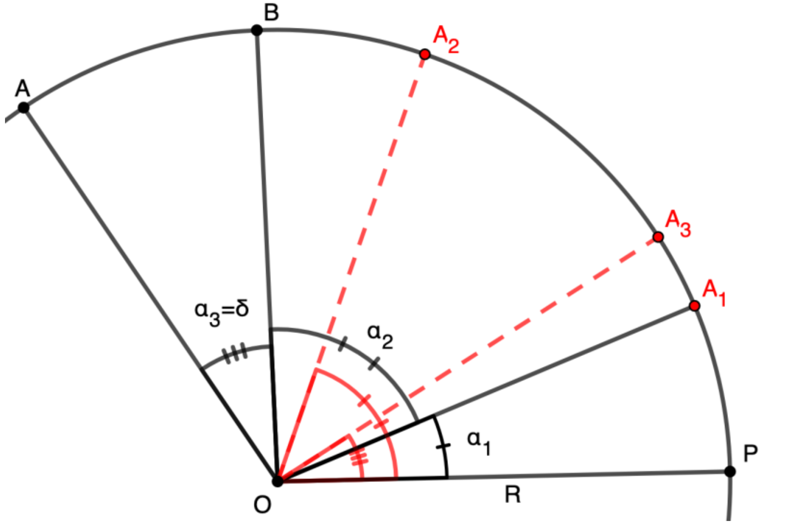


THEOREM 1

- Fix a P point on the circle, and for every point Q let the central angle be POQ∡.



It is sufficient to properly k-color the points whose central angle has a residual D for each D square-free integer.



Theorem 2

For $D \neq D_r$ the graph G_D is 2-colorable, where G_D is the subgraph of G^r induced by the points with residual D, and $r = p\sqrt{D_R}$.

Theorem 3

Given a circle of radius $\frac{a}{b}\sqrt{D_R}$, the subgraph of G^{odd} on the circle G_{D_R} is trivial, if

- $2 \nmid$ b and $D_R \not\equiv 3$ or $7 \mod 8$
- $2 \mid$ b but $4 \nmid$ b and $D_R \not\equiv 1$ or $5 \mod 8$

Rational linear relations of geodetic angles

- Motivation: Finding a way to generate more examples for later uses
- Def: α is a pure geodetic angle, if $\sin^2\alpha$ is rational
- Def: α is a mixed godetic angle, if it's a linear combination of pure geodetic angles with rational coefficients
- The mixed geodetic angles form a vector space over $\mathbb{Q}$ $\Rightarrow$ there exists a basis
- Basis: We express $4p^s$ as a^2+db^2 for the smallest possible positive s. Then:

$$\langle p \rangle_d = \frac{1}{s} \sin^{-1} \sqrt{\frac{db^2}{4p^s}}$$

Theorem 4 and 5

- Every pure geodetic angle is uniquely expressible as a rational multiple of π plus an integral linear combination of the angles $\langle p \rangle_d$. So the angles $\langle p \rangle_d$ supplemented by π, form a basis for the space of mixed geodetic angles.
- If $\tan\theta = \frac{b}{a}\sqrt{d}$ for integers a, b, d with square-free positive d and with relatively prime a and b, and if the prime factorization of $a^2 + b^2 d$ is $p_1 p_2 \ldots p_n$ (including multiplicity), then we have

$$\theta = t\pi \pm \langle p_1 \rangle_d \pm \langle p_2 \rangle_d \pm \ldots \langle p_n \rangle_d$$

for some rational t.

- $\tan \alpha = \frac{b}{a}\sqrt{d}$ ⇒argument of $a + b\sqrt{-d}$
- Problem: not necessarily a unique factorization domain
- BUT the ideals have unique factorisation
- Argument:

$$\arg(I) = \begin{cases} \arg(\kappa), \text{ if I} = (\kappa) \text{ principal} \\ \frac{1}{s}\arg(I^s), \text{ if I is not principal} \end{cases}$$

- This is unique up to multiplication by a unit

DEFINING THE $\langle p \rangle_d$ VALUES

- p rational prime, (p) can:
  - remain prime
  - ramify
  - split
- If splits: let $\langle p \rangle_d$ be the argument of one of the prime ideals, such that
  - $0 < \langle p \rangle_d < \frac{\pi}{2}$, if the factors of (p) are principal
  - $0 < \langle p \rangle_d < \frac{\pi}{4}$, if not.

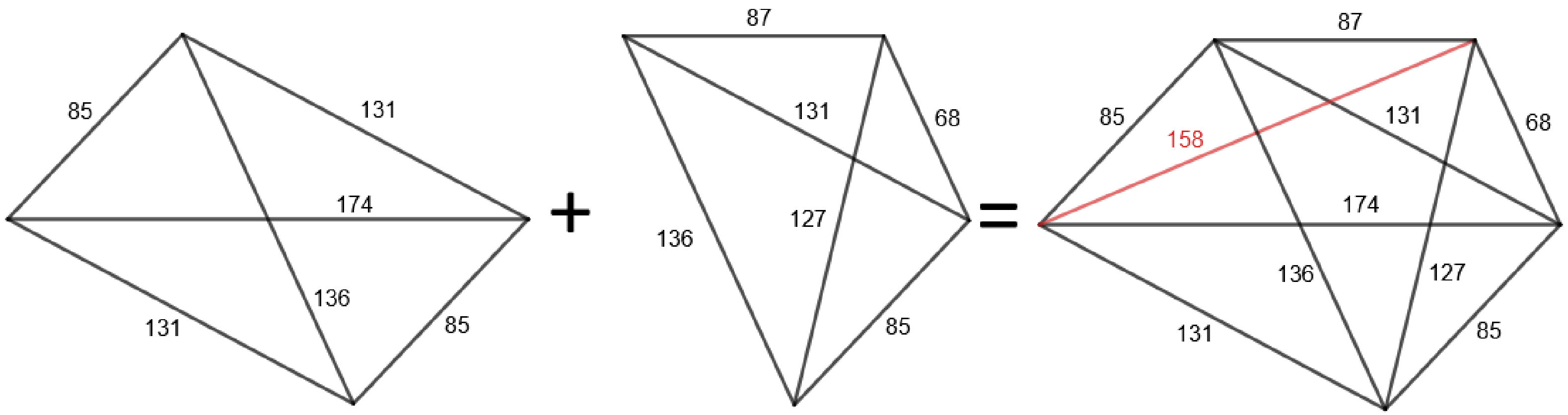
- $\tan\alpha = \frac{b}{a}\sqrt{d} \Rightarrow \mathrm{I} = \left(\mathrm{a} + \mathrm{b}\sqrt{-d}\right) = \pi_1\pi_2 \ldots \pi_n$
- $(p_1)(p_2) \ldots (p_n) = (a^2 + b^2 d) = I\bar{I} = \pi_1\bar{\pi}_1\pi_2\bar{\pi}_2 \ldots \pi_n\bar{\pi}_n$
- $\pi_i\bar{\pi}_i=(p_i)$

DIFFICULTIES IN THE IMPLEMENTATION

- The (2) doesn’t behave the as described above, when remains prime
- The coefficient of π
- Inaccuracies

APPLICATIONS

- Generating examples
- Search for a bigger integer sided K_n:



Uses of AI in the project

- Debugging the code
- Generating examples in various cases

FURTHER RESEARCH

- Moving forward, I'd like to refine the code, fix bugs, handle edge cases, and generally make it cleaner, more efficient, and more user-friendly for the future.

Thank you for listening!