

# (p,q)-Type Theorems in Geometric Settings

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## 1 Introduction

Continuing with the reading project started in the previous semester, I studied some new literature on piercing problems of finite families of convex sets satisfying the  $(p, q)$ -condition which was originally raised by Hadwiger and Debrunner.<sup>1</sup> It is known that such families can be pierced with finitely many points.

**Theorem 1.1** (The  $(p, q)$ -theorem, Alon-Kleitman, 1992). *Let  $p, q, d$  be integers with  $p \geq q \geq d + 1$ . Then there exists a number  $HD_d(p, q)$  such that if  $\mathcal{F} \subseteq \mathbb{R}^d$  is a family of convex sets satisfying the  $(p, q)$ -condition, then  $\mathcal{F}$  has a transversal consisting of at most  $HD_d(p, q)$  points.*

However, the general argument often gives huge bounds for the number  $HD_d(p, q)$ . The main focus in the previous semester was to understand the proof of the current best bound for the case  $p = 4, q = 3$  given by McGinnis [8], whose approach relies on two main results.

**Theorem 1.2** (KKM theorem [5]). *Let  $A_1, \dots, A_n$  be open subsets of the simplex  $\Delta^{n-1}$  such that for every face  $\sigma$  of  $\Delta^{n-1}$  we have  $\sigma \subseteq \bigcup_{e_i \in \sigma} A_i$ . Then  $\bigcap_{i=1}^n A_i \neq \emptyset$ .*

The second theorem used in that proof is a result of Tardos [11] concerning 2-intervals. Let  $L_1, \dots, L_k$  be  $k$  homeomorphic copies of  $\mathbb{R}$ . A  $k$ -interval is a set of the form  $I = I_1 \cup \dots \cup I_k$ , where each  $I_i$  is an interval in  $L_i$ .

**Theorem 1.3** (Tardos [11]). *If  $\mathcal{I}$  is a family of 2-intervals, then  $\tau(\mathcal{I}) \leq 2\nu(\mathcal{I})$ .*

As part of the work carried out during this semester, I explored some of the currently known colorful variants for the theorems studied before, as well as a KKM-type argument used to show a better bound for the piercing number in the case of discrete rectangles [9].

## 2 The colorful $(p, q)$ -problem

To begin with, we extend the  $(p, q)$ -property to a colorful setting. Let  $\mathcal{F}_1, \dots, \mathcal{F}_p$  be finite families of convex sets in  $\mathbb{R}^d$ , and write  $\mathcal{F} = \bigcup_i \mathcal{F}_i$ . A heterochromatic  $p$ -tuple is a tuple  $C_1, \dots, C_p$  with  $C_i \in \mathcal{F}_i$ . The family  $\mathcal{F}$  satisfies the heterochromatic  $(p, q)_H$ -condition, if every heterochromatic  $p$ -tuple of  $\mathcal{F}$  contains an intersecting  $q$ -tuple.

<sup>1</sup>A family  $\mathcal{F}$  of sets in  $\mathbb{R}^d$  satisfies the  $(p, q)$ -condition if among any  $p$  sets in  $\mathcal{F}$ , there exist  $q$  sets with a common point.

**Theorem 2.1** (Heterochromatic  $(p, q)$ -theorem [2]). *Let  $p, q, d$  be integers with  $p \geq q \geq d+1$ . There exists a constant  $M(p, q, d)$  such that the following holds: if  $\mathcal{F}_1, \dots, \mathcal{F}_p$  are finite families of convex sets satisfying the heterochromatic  $(p, q)_H$ -condition, then for at least  $q - d$  indices  $i \in [p]$  we have  $\tau(\mathcal{F}_i) \leq M(p, q, d)$ .*

The proof of this theorem follows the same spirit as the proof of the original  $(p, q)$ -theorem. In fact, one needs a colorful version of the fractional Helly theorem as well.

**Lemma 2.1.1** (Colorful Fractional Helly Theorem [2]). *Let  $\mathcal{F}_1, \dots, \mathcal{F}_{d+1}$  be finite families of convex sets in  $\mathbb{R}^d$ ,  $\mathcal{F} = \bigcup \mathcal{F}_i$  and let  $\alpha \in (0, 1)$ . If  $\alpha |\mathcal{F}_1| \cdots |\mathcal{F}_{d+1}|$  heterochromatic  $(d+1)$ -tuples intersect, then there is an  $i \in [d+1]$  such that  $\mathcal{F}_i$  contains an intersecting subfamily of size  $\frac{\alpha}{d+1} |\mathcal{F}_i|$ .*

*Proof.* (Taken from [2]) Let  $\mathcal{H}$  be the  $(d+1)$ -partite hypergraph with class  $i$  identified with  $\mathcal{F}_i$  and edges  $e \in \mathcal{H}$  corresponding to intersecting heterochromatic  $(d+1)$ -tuples of  $\mathcal{F}$ . Thus  $e$  is simply  $(C_1, \dots, C_{d+1})$  with  $C_i \in \mathcal{F}_i$  and  $\bigcap_1^{d+1} C_i \neq \emptyset$ . Set  $C(e) = \bigcap_1^{d+1} C_i$ . Define a partial edge as  $f = (C_1, \dots, C_{i-1}, C_{i+1}, \dots, C_{d+1})$  if the intersection,  $C(f)$ , of these  $d$  convex sets is nonempty. Assume as we may that all  $C \in \mathcal{F}$  are polytopes. Then all  $C(e)$  and  $C(f)$  are polytopes as well, and we can choose a vector  $a \in \mathbb{R}^d$  so that the minimum of the scalar product  $a \cdot x$  over all  $x$  in  $C(e)$  and the minimum over all  $x$  in  $C(f)$  is reached at unique points  $x(e)$  and  $x(f)$ .

Claim: For every  $e \in \mathcal{H}$  there is a partial edge  $f \subset e$  with  $x(e) = x(f)$ .

*Proof.* Let  $H = \{x \in \mathbb{R}^d : ax < ax(e)\}$ , this is an open halfspace and the definition of  $x(e)$  implies that

$$H \cap C(e) = H \cap C_1 \cap \cdots \cap C_{d+1} = \emptyset.$$

So these  $d+2$  convex sets have empty intersection. By Helly's theorem some  $d+1$  of them have empty intersection. This  $(d+1)$ -tuple cannot be  $C_1, \dots, C_{d+1}$  so it is of the form  $H, C_1, \dots, C_{i-1}, C_{i+1}, \dots, C_{d+1}$  for some  $i$ . This means that  $\bigcap_{j \neq i} C_j$  is disjoint from  $H$ . But it contains  $x(e)$  so  $x(f) = x(e)$  with  $f = (C_1, \dots, C_{i-1}, C_{i+1}, \dots, C_{d+1})$ .  $\square$

Now let  $N_i = |\mathcal{F}_i|$  for all  $i$  and let  $N = N_1 \dots N_{d+1}$ . Write  $\mathcal{H}_i$  for the  $d$ -partite hypergraph whose edges are the partial edges  $f$  missing class  $i$ . Clearly,  $|\mathcal{H}_i| \leq N/N_i$ . For  $f \in \mathcal{H}_i$  let  $\mathcal{F}_i(f) = \{C \in \mathcal{F}_i : x(f) \in C\}$ . Note that  $\mathcal{F}_i(f)$  is an intersecting subfamily of  $\mathcal{F}_i$ . We define  $\alpha_i$  by

$$\alpha_i N_i = \max_{f \in \mathcal{H}_i} |\mathcal{F}_i(f)|$$

Finally, we double count the pairs  $(e, f)$  with  $e \in \mathcal{H}$ ,  $f \subset e$ ,  $f \in \mathcal{H}_i$  for some  $i$ , and  $x(e) = x(f)$ . The claim says that the number of such pairs is at least  $\alpha N_1 \dots N_{d+1} = \alpha N$ . Hence

$$\begin{aligned}
\alpha N &\leq \text{number of such pairs } (e, f) \\
&= \sum_{i=1}^{d+1} \sum_{f \in \mathcal{H}_i} \text{number of } e \in \mathcal{H} \text{ with } (e, f) \text{ being such a pair} \\
&\leq \sum_{i=1}^{d+1} \sum_{f \in \mathcal{H}_i} |\{C \in \mathcal{F}_i : x(f) \in C\}| \leq \sum_{i=1}^{d+1} \sum_{f \in \mathcal{H}_i} \alpha_i N_i \\
&\leq \sum_{i=1}^{d+1} \alpha_i N_i \frac{N}{N_i} = \sum_{i=1}^{d+1} \alpha_i N
\end{aligned}$$

This implies that  $\alpha \leq \sum_{i=1}^{d+1} \alpha_i$  and so  $\alpha_i \geq \frac{\alpha}{d+1}$  for some  $i$ .  $\square$

Moreover, this lemma also allows us to prove a fractional version for the heterochromatic  $(p, q)$ -theorem.

**Theorem 2.2** (Fractional Heterochromatic  $(p, q)$ -theorem [2]). *Let  $\alpha > 0$  and  $p, d$  be integers with  $p \geq d + 1$ . There exists a constant  $\gamma(\alpha, p, d) > 0$  such that given finite families  $\mathcal{F}_1, \dots, \mathcal{F}_p \subseteq \mathbb{R}^d$  of convex sets satisfying the  $(p, d + 1)_H$ -condition for  $(1 - \alpha)|\mathcal{F}_1| \cdots |\mathcal{F}_p|$  of the heterochromatic  $p$ -tuples of  $\mathcal{F}$ , then there exists  $i \in [p]$  such that  $\mathcal{F}_i$  contains an intersecting subfamily of size at least  $\gamma|\mathcal{F}_i|$ .*

### 3 Piercing Discrete Rectangles

If we now restrict to families of boxes, then one can prove similar theorems. The first natural one is a Helly theorem. The trace of a family  $\mathcal{F}$  on a set  $X$ , denoted by  $\mathcal{F}|_X$ , is defined as  $\mathcal{F}|_X := \{F \cap X : F \in \mathcal{F}\}$ .

**Theorem 3.1** (Halman's theorem; Theorem 2.10 in [3]). *Let  $d$  be a positive integer. Let  $P$  be a finite set in  $\mathbb{R}^d$ , and let  $\mathcal{B}$  be a finite family of boxes in  $\mathbb{R}^d$ . If for every subfamily  $\mathcal{B}' \subseteq \mathcal{B}$  of size at most  $2d$  the trace  $\mathcal{B}'|_P$  is intersecting, then  $\mathcal{B}|_P$  is also intersecting.*

Moreover, I studied a  $(p, q)$ -type theorem for these families and an argument due to Wei Rao for obtaining a bound on the number of points needed to pierce discrete boxes in the case  $p = q = d = 2$  which makes use of yet another KKM-type theorem.

**Theorem 3.2** (KKMS theorem [6]). *Let  $Q$  be a polytope, and suppose that for every face  $F$  of  $Q$  a point  $q(F) \in F$  is chosen. Let  $B_F$  be an open subset of  $Q$  assigned to each face  $F$ , such that every face  $F'$  of  $Q$  satisfies  $F' \subseteq \bigcup_{F \subseteq F'} B_F$ . Then there exists a collection  $\mathcal{F}$  of faces of  $Q$  such that  $q(Q) \in \text{conv}\{q(F) : F \in \mathcal{F}\}$  and  $\bigcap_{F \in \mathcal{F}} B_F \neq \emptyset$ .*

**Theorem 3.3** (Theorem 3.1 in [9]). *Let  $P \subseteq \mathbb{R}^2$ . Suppose that  $\mathcal{B}$  is a finite family of rectangles in  $\mathbb{R}^2$  such that the intersection of any two rectangles contains a point of  $P$ . Then there exists a subset  $S \subseteq P$  of size at most 8 such that  $B \cap S \neq \emptyset$  for every  $B \in \mathcal{B}$ .*

It is worth mentioning that, in the proof of this theorem, one can simplify the argument and avoid using the KKMS theorem by making use of a generalization of Tardos' theorem for 4-intervals.

**Theorem 3.4** (Theorem 1.4 in [4]). *Let  $I$  be a finite family of  $k$ -intervals with  $k \geq 2$ . Then  $\tau(I) \leq (k^2 - k)\nu(I)$ .*

But in this case the bound we get is 12 instead of the bound 8 obtained by the method proposed by Wei Rao. For the general case of boxes contained in  $\mathbb{R}^d$  we have the following theorem.

**Theorem 3.5** (Theorem 5 in [10]). *Let  $p, q, d$  be positive integers with  $p \geq q \geq 2$ . There exists  $N := N(p, q, d)$  such that for any finite set  $P \subseteq \mathbb{R}^d$  and any finite family  $\mathcal{B}$  of boxes in  $\mathbb{R}^d$ , the following holds. If the trace  $\mathcal{B}|_P$  has the  $(p, q)$ -property and does not contain the empty set, then  $\tau(\mathcal{B}|_P) \leq N$ .*

## 4 Future work

I am interested in continuing to explore the literature on  $(p, q)$ -type problems, as well as trying to extend these arguments to colorful variants. For the particular case of the  $(4, 3)$ -problem, I would also like to consider families of halfplanes and other special families, as in [7].

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