# (p,q)-Type Theorems in Geometric Settings

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# The general problem

A family $\mathcal{F}$ of sets in $\mathbb{R}^d$ satisfies the $(p, q)$-condition if among any $p$ sets in $\mathcal{F}$, there exist $q$ sets with a common point.

**Theorem (The $(p, q)$-theorem)**

Let $p, q, d$ be integers with $p \geq q \geq d + 1$. Then there exists a number $HD_d(p, q)$ such that if $\mathcal{F} \subseteq \mathbb{R}^d$ is a finite family of convex sets satisfying the $(p, q)$-condition, then $\mathcal{F}$ has a transversal consisting of at most $HD_d(p, q)$ points.

# The $(4, 3)-$ problem

However, the general argument often gives huge bounds for the number $HD_d(p, q)$. The main focus in the previous semester was to understand the proof of the current best bound for the case $p = 4$, $q = 3$ given by McGinnis [2], whose approach relies on two main results.

# The $(4,3)-$ problem

**Theorem (KKM theorem)**

*Let* $A_1, \ldots, A_n$ *be open subsets of the simplex* $\Delta^{n-1}$ *such that for every face* $\sigma$ *of* $\Delta^{n-1}$ *we have* $\sigma \subseteq \bigcup_{e_i \in \sigma} A_i$*. Then* $\bigcap_{i=1}^{n} A_i \neq \emptyset$*.*

The second theorem used in that proof is a result of G. Tardos concerning 2-intervals. Let $L_1, \ldots, L_k$ be $k$ homeomorphic copies of $\mathbb{R}$. A *k-interval* is a set of the form $I = I_1 \cup \cdots \cup I_k$, where each $I_i$ is an interval in $L_i$.

**Theorem (Tardos)**

*If* $\mathcal{I}$ *is a family of* 2*-intervals, then* $\tau(\mathcal{I}) \leq 2\nu(\mathcal{I})$*.*

# The colorful $(p, q)$-problem

Let $\mathcal{F}_1, \ldots, \mathcal{F}_p$ be finite families of convex sets in $\mathbb{R}^d$, and write $\mathcal{F} = \bigcup_i \mathcal{F}_i$. A heterochromatic $p$-tuple is a tuple $C_1, \ldots, C_p$ with $C_i \in \mathcal{F}_i$. The family $\mathcal{F}$ satisfies the heterochromatic $(p, q)_H$-condition, if every heterochromatic $p$-tuple of $\mathcal{F}$ contains an intersecting $q$-tuple.

**Theorem (Heterochromatic $(p, q)$-theorem [2])**

*Let $p, q, d$ be integers with $p \geq q \geq d+1$. There exists a constant $M(p, q, d)$ such that the following holds: if $\mathcal{F}_1, \ldots, \mathcal{F}_p$ are finite families of convex sets satisfying the heterochromatic $(p, q)_H$-condition, then for at least $q - d$ indices $i \in [p]$ we have $\tau(\mathcal{F}_i) \leq M(p, q, d)$.*

# Idea of the proof

**Lemma (Colorful Fractional Helly Theorem [2])**

*Let $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ be finite families of convex sets in $\mathbb{R}^d$, $\mathcal{F} = \bigcup \mathcal{F}_i$ and let $\alpha \in (0,1)$. If $\alpha|\mathcal{F}_1| \cdots |\mathcal{F}_{d+1}|$ heterochromatic $(d+1)$-tuples intersect, then there is an $i \in [d+1]$ such that $\mathcal{F}_i$ contains an intersecting subfamily of size $\frac{\alpha}{d+1}|\mathcal{F}_i|$.*

# Piercing Discrete Rectangles

The trace of a family $\mathcal{F}$ on a set $X$, denoted by $\mathcal{F}|_X$, is defined as $\mathcal{F}|_X := \{F \cap X : F \in \mathcal{F}\}$.

**Theorem (Halman's theorem; Theorem 2.10 in [3])**

*Let $d$ be a positive integer. Let $P$ be a finite set in $\mathbb{R}^d$, and let $\mathcal{B}$ be a finite family of boxes in $\mathbb{R}^d$. If for every subfamily $\mathcal{B}' \subseteq \mathcal{B}$ of size at most $2d$ the trace $\mathcal{B}'|_P$ is intersecting, then $\mathcal{B}|_P$ is also intersecting.*

# Piercing Discrete Rectangles

**Theorem (KKMS theorem [1])**

*Let $Q$ be a polytope, and suppose that for every face $F$ of $Q$ a point $q(F) \in F$ is chosen. Let $B_F$ be an open subset of $Q$ assigned to each face $F$, such that every face $F'$ of $Q$ satisfies $F' \subseteq \bigcup_{F \subseteq F'} B_F$. Then there exists a collection $\mathcal{F}$ of faces of $Q$ such that $q(Q) \in conv\{q(F) : F \in \mathcal{F}\}$ and $\bigcap_{F \in \mathcal{F}} B_F \neq \emptyset$.*

**Theorem (Theorem 3.1 in [3])**

*Let $P \subseteq \mathbb{R}^2$. Suppose that $\mathcal{B}$ is a finite family of rectangles in $\mathbb{R}^2$ such that the intersection of any two rectangles contains a point of $P$. Then there exists a subset $S \subseteq P$ of size at most 8 such that $B \cap S \neq \emptyset$ for every $B \in \mathcal{B}$.*

**Theorem (Theorem 1.4 in [4])**

*Let $I$ be a finite family of $k$-intervals with $k \geq 2$. Then $\tau(I) \leq (k^2 - k)\nu(I)$.*

For the general case of boxes contained in $\mathbb{R}^d$ we have the following theorem.

**Theorem (Theorem 5 in [4])**

*Let $p, q, d$ be positive integers with $p \geq q \geq 2$. There exists $N := N(p, q, d)$ such that for any finite set $P \subseteq R^d$ and any finite family $\mathcal{B}$ of boxes in $\mathbb{R}^d$, the following holds. If the trace $\mathcal{B}|_P$ has the $(p, q)$-property and does not contain the empty set, then $\tau(\mathcal{B}|_P) \leq N$.*

# Future work

I am interested in continuing to explore the literature on $(p, q)$-type problems, as well as trying to extend these arguments to colorful variants. For the particular case of the $(4, 3)$-problem, I would also like to consider families of halfplanes and other special families.

# References

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AI tools were used to find relevant literature only.