Random walks on graphs and electric networks

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1 Electric networks and random walks

We start by describing a connection between random walks on countable graphs and electric networks. A graph can be viewed as a network, whose edges are wires between the vertices. We do not assume the random walk to be simple: we might choose neighbors of a vertex with different probabilities. The transition probabilities between two vertices tell us how easy it is for an electron to travel through that edge compared to other edges with the same beginning. The corresponding physical property is the conductance of wires, which is the inverse of the resistance.
In this paper every graph or Markov chain is countable unless explicitly stated otherwise. At this point we must point out that this analogy only works when the walk is reversible, because the conductance of a cable is clearly the same regardless of which way a current goes through it.

Definition 1.1. A random walk on a graph $G = (V, E)$, or more generally a Markov chain with state space V and transition probabilities $p(x, y)$ is reversible if there exists a measure π on the vertices such that $\pi(x)p(x, y) = \pi(y)p(y, x)$ for every $(x, y) \in E$. π is called the stationary measure of the random walk.

Remark 1.2. Since Markov chains and graphs with transition probabilities are basically the same we will use the notion which is more convinient.

Remark 1.3. We must observe that π does not have to be a probability measure on V. For example look at $\mathbb{Z}$ with the simple random walk. In this case π is the counting measure.

Remark 1.4. Note that π is unique up to multiplying with a positive constant as long as G is connected.

It is clear that the only way our analogy can work is if we define the conductance $c : E \longrightarrow \mathbb{R}_{\geq 0}$ of the edges from the reversible random walk as $c(x, y) = c(y, x) = \pi(x)p(x, y)$.

Conversely, if we are given the conductances then the only way to define transition probabilities is by $p(x,y) = \dfrac{c(x,y)}{\sum_{x\sim y'} c(x,y')}$. The function $\pi : V \longrightarrow \mathbb{R}_{\geq 0}$ witnessing reversibility must be $\pi(x) = \sum_{x\sim y} c(x,y)$. Let us use this correspondence to help answer questions about random walks.

An important question regarding infinite graphs is recurrence and transience. That is, determining whether the random walk started at an arbitrary vertex returns there with probability 1, or strictly less than 1. In the case of finite graphs what we are interested in is that given two disjoint subsets A and B of V what is the probabilty that the random walk from $x \in V$ reaches A before B. The results in the finite case help in distinguishing recurrence and transience in infinite graphs. We use the notation $\mathbb{P}_x[\tau_A < \tau_B]$ to denote the probability above, where τ_A is the first time the random walk reaches A. The standard way of computing these probabilities is to view them as a function $f(x) = \mathbb{P}_x[\tau_A < \tau_B]$ on V, and observing the fact that f is harmonic outside of $A \cup B$:

$$f(x) = \mathbb{P}_x[\tau_A < \tau_B] = \sum_y \mathbb{P}_x[\text{first step is to } y] \cdot \mathbb{P}_x[\tau_A < \tau_B \mid \text{first step is } y] = \sum_y p(x,y) f(y).$$

The function f need not be harmonic on $A \cup B$, the argument above fails, because there is no first step.

Let us present a few additional results about harmonic functions and then reap the benefits of using electric networks.

Definition 1.5. Let G be a graph with transition probabilities $p(x,y)$ and $W \subseteq V$. The graph absorbed off of W is the subgraph we get by removing the edges between vertices outside W.

Theorem 1.6 (**Maximum principle**). *Let G be a graph and $W \subseteq V$. Then if $f : V \longrightarrow \mathbb{R}$ is harmonic on W and the supremum of f is achieved on some $x_0 \in W$ then f is constant on those vertices which can be reached from x on the graph absorbed off of W*

Theorem 1.7 (**Uniqueness principle**). *Let W be a finite proper subset of V. If $V \setminus W$ can be reached from every point of W in the graph absorbed off of W and $f, g : V \longrightarrow \mathbb{R}$ are harmonic functions on W and equal outside W i.e. $f|_{V\setminus W} \equiv g|_{V\setminus W}$, then $f \equiv g$.*

The uniqueness principle means that if we know the values of f where it is not harmonic we know the function. The next question is whether there always exists such an f.

Theorem 1.8 (**Existence principle**). *Let W be a proper subset of V and $f_0 : V \setminus W \longrightarrow \mathbb{R}$ a bounded function. Then there exists an $f : V \longrightarrow \mathbb{R}$ which is harmonic on W and $f|_{V\setminus W} = f_0$.*

We have seen above that $\mathbb{P}_x[\tau_A < \tau_B]$ is harmonic outside of $A \cup B$ when A and B are disjoint. Now we bring in a new concept, the potential function $v : V \longrightarrow \mathbb{R}$. Suppose that we have an electric network with conductances $c(x,y)$ and we assign a potential v_0 to some subset W of the vertices. We can picture it as external devices, for example batteries, connected in a way that these points of the system are kept at this fixed potential. The potential function is the harmonic extension v of v_0 to V. Usually $W = A \cup Z$ where $v|_A$ is some $v_A > 0$ constant and $v|_Z \equiv 0$. Then $\mathbb{P}_x[\tau_A < \tau_B] = \frac{v(x)}{v_A}$. Most of the time $A = \{a\}$ is a single point and $v_A = 1$.

Using the potential function we can define the associated current function i on the edges by

$$i(x,y) = c(x,y)(v(x) - v(y)),$$

the same as if we were in physics and working with real cables, networks and conductances. A trivial observation is that $i(x,y) = -i(y,x)$ and the current flows from the higher potential towards the lower. Furthermore, if v is harmonic at x then

$$\sum_y i(x,y) = \sum_y c(x,y)(v(x) - v(y)) = v(x)\sum_y c(x,y) - \sum_y c(x,y)v(y) = 0.$$

This means that i is a flow between A and Z.

The formula for the current is called Ohm's law in physics, with the resistance function r defined as $r = \frac{1}{c}$. The fact that i is a flow is called Kirchhoff's node law.

Ohm's law: If $x \sim y$ are vertices in V then the current satisfies

$$\frac{v(x) - v(y)}{i(x,y)} = r(x,y)$$

.

Using these properties we can derive **Kirchhoff's cycle law**, which says that if $e_1, \ldots, e_n$ is a cycle then

$$\sum_{i=1}^{n} i(e)r(e) = 0.$$

2 Effective conductance and resistance

Let us go back to our original problem, i.e. determining $\mathbb{P}_x[\tau_A < \tau_Z]$, and assume that $A = \{a\}$. Now we will try to calculate the probability of the random walk starting at a going back to a before reaching Z. Let τ_a^+ be the first nonzero time the random walk hits a. We aim to determine $\mathbb{P}[a \to Z] := \mathbb{P}_a[\tau_Z < \tau_a^+]$ using the tools introduced in the previous chapter.

Recall that π was the stationary measure of the random walk. Let the potential be $v(a)$ at a and 0 at Z. Using $\mathbb{P}_x[\tau_a < \tau_Z] = \frac{v(x)}{v(a)}$ we get that

$$\mathbb{P}[a \to Z] = \mathbb{P}_a[\tau_Z < \tau_a^+] = \sum_{a \sim x} p(a,x)\mathbb{P}_x[\tau_Z < \tau_a] = \sum_{a \sim x} \frac{c(a,x)}{\pi(a)}(1 - \mathbb{P}_x[\tau_a < \tau_Z]) =$$

$$= \sum_{a\sim x} \frac{c(a,x)}{\pi(a)} \left(1 - \frac{v(x)}{v(a)}\right) = \frac{1}{v(a)\pi(a)} \sum_x c(a,x)(v(a) - v(x)) = \frac{1}{v(a)\pi(a)} \sum_x i(a,x).$$

In physics, the effective conductance of the network between a and Z is computed as $C_{eff} = \frac{\text{strength of the current}}{\text{difference of the potentials}}$ which means that

$$C_{eff} = \frac{\sum_{a\sim x} i(a,x)}{v(a)} = \pi(a)\mathbb{P}[a \to Z].$$

So let us define C_{eff} as $\pi(a)\mathbb{P}[a \to Z]$. We use the notation $\mathcal{C}(a \leftrightarrow Z)$ for the effective conductance.

Intuitively by substituting ∞ for Z in an infinite graph $\mathcal{C}(a \leftrightarrow \infty)$ being positive or not detects whether a graph is transient or not. Therefore a graph is transient if current flows from a to ∞ which can only flow if the effective conductance is bigger than 0 which is the same as the effective resistance being finite. In practice we will take $a \in G_1 \subseteq G_2 \subseteq \dots$ where $\bigcup G_n = G$ and look for the limit of $\mathcal{C}(a \leftrightarrow V \setminus V(G_n))$.

Consequently, we can determine recurrence of infinite graphs if we know the effective conductance from a to infinity. For example $\mathbb{Z}$ with the simple random walk corresponds to the network with constant 1 conductances. This means the resistances are also 1, and, as the edges are connected "in series", the resistances add up to an infinite effective resistance between 0 and infinity. This gives us that the effective conductance is zero, so the walk is recurrent.

Generally it is not that easy to calculate the effective resistance but it is often easier than determining transience or recurrence probabilistically. It can be proven that the laws about connecting wires in parallel or in series are true in this setting also.

Using these results it is pretty straightforward to show that Z^2 is also recurrent. Let E_n denote the edges with exactly one endvertex in the box $[-n,n]^2$. Then E_n are disjoint vertex cuts separating the origin from infinity, with size linear in n. We can lower bound the effective resistance by contracting the perimeter of each box. Edges in the same cut become parallel, hence the resistance is of order $1/n$. But the cuts themselves are connected in series, so the effective resistance is infinite. This argument can be generalized to any graph with disjoint finite cuts, see the so-called Nash-Williams criterion in [1, Chapter 2].

3 Uniform spanning trees

In the previous chapter we saw that random walks and electric networks are closely related. Now in this chapter we present the connections between random walks and random spanning trees.

The main idea is that the probability of an edge being in a random spanning tree is related to the transition probability through that edge. We will consider directed orientations because in an undirected graph every edge has two transition probabilities. On a directed graph we call a subgraph a spanning tree if forgetting the orientation it is a spanning tree. Furthermore, there exists a vertex called the root such that every edge points towards that vertex.

Wilson in [2] devised an effecient algorithm for creating uniform spanning trees in 1996. Given a finite graph G, in the first step we choose a vertex a to be the root. Let T_0 be $\{a\}$, the tree with one vertex and zero edges. The general step is the following. We choose a vertex a_i in $V \setminus T_i$ and start the random walk at a_i. When we first reach T_i we stop the walk, erase the loops, and add the remaining edges and vertices of the walk to T_i, getting T_{i+1}. The algorithm stops when there are no more edges in $V \setminus T_i$.

Let us observe that the algorithm works on recurrent infinite graphs. In a recurrent graph, the probability of returning to a vertex a is 1. Since G is connected, we reach any vertex x with positive probability. By combining these facts we get that the probability of reaching a from x must be 1. So Wilson's algorithm terminates on recurrent graphs.

Remark 3.1. It can be proven that the distribution obtained from Wilson's algorithm does not depend on which verteces we choose to be the a_i-s.

4 Further applications of networks

By introducing the concept of energy we can show that transience is monotone i.e. if we increase the conductances of a transient graph, it remains transient. Furthermore a graph is transient if there exists a current with finite energy from a to infinity.

We can find many properties of weighted uniform and fractional spanning trees using conductances of edges and networks. We can also find maximal entropy weighted uniform spanning trees using Wilson's algorithm.

Next semester we are hoping to exploit the electric network framework in cycle matroids of graphings and investigate if fractional spanning forests of a graphing are convex combinations of $\{0, 1\}$-valued spanning forests, in the special case when the graphing is hyperfinite. This question was studied by Bérczy, Borbényi, Lovász and Tóth in [3], but was only resolved partially.

5 Declaration

I, the undersigned, Gergely Szőke, hereby declare that during the preparation of this report I used the AI-based tools listed below for the completion of the following tasks:

Task	Tool Used	Area of Use
Grammar and language checking	ChatGPT	Entire report
LaTeX syntax and formatting advice	ChatGPT	Entire report

References

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[2] James Gary Propp and David Bruce Wilson, *How to Get a Perfectly Random Sample from a Generic Markov Chain and Generate a Random Spanning Tree of a Directed Graph*, Journal of Algorithms, Vol. 27, No. 2, 1998, pp. 170–217.
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[3] Kristóf Bérczy, Márton Borbényi, László Lovász, and László Tóth, *Cycle Matroids of Graphings: From Convergence to Duality*, Combinatorica, Vol. 45, 2025, pp. 1–32.
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