

Directed Studies 2

On the Values of Landau's Function

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Introduction

S_n , the symmetric group over n elements has the order of $n!$, while the maximal order which any of its elements can have is way smaller. We will denote the latter function by $g(n)$, it can be found as Landau's function in literature, named after Edmund Landau (1877-1938), who (in 1902) proved that: $\lim_{n \rightarrow \infty} \frac{\ln(g(n))}{\sqrt{n \ln(n)}} = 1$.

We will show that the sequence $g(n)$ has a threshold N_m for every m such that $m|g(n) : \forall n > N_m$, the special case $N_2 = 15$.

Proof for $m=2$

As the easiest example, we start, with the case when $m = 2$, to show our method. Let's understand a bit of how $g(n)$ works. The order of a permutation as an element of S_n is nothing else but the least common multiple, of its cycle lengths, while the sum of its cycle lengths is exactly n . So we want the least common multiple of some integers to be maximal, while their sum is bounded (we can always supplement the sum with ones). If $2 \nmid g(k)$, then $g(k+2) \geq 2g(k)$, because we can use the exact same set of numbers (which have least common multiple equal to $g(k)$, while sum $\leq k$), completed with a 2. With this we reach a contradiction because the derivative of the log-asymptotic growth converges to 0, namely:

$$\log(g(n)) \approx \sqrt{n \ln(n)} \text{ and } \left(\sqrt{x \ln(x)} \right)' = \frac{\ln(x) + 1}{2\sqrt{x \ln(x)}} \text{ and } \lim_{n \rightarrow \infty} \frac{\ln(x) + 1}{2\sqrt{x \ln(x)}} \rightarrow 0.$$

□

Proof for general integer and comments

We can do essentially the same, because although adding m doesn't necessarily mean $g(k+m) \geq mg(k)$, but it still has to be \geq a multiple of $g(k)$, meaning $g(k+m) \geq 2g(k)$, which would for any m still imply exponential growth (more precisely $\log(g(k+m)) \geq \log(g(k)) + \log(2)$, but $\sqrt{n \log(n)}$ converges to 0), thus we have a proof for the existence of N_m for any m . The nature of this proof is quite far from a constructive one, since in the article, that I read, the log-asymptotic growth was proven by looking at somewhat of the greedy algorithmic version of $g(n)$, denoted by $f(n)$, meaning $f(n) := \prod_{i=1}^s p_i$, where $\sum_{i=1}^s p_i \leq n$, but $\sum_{i=1}^{s+1} p_i > n$, while p_i are the prime numbers in ascending order, then invokes prime number theorem. From this proof, it follows, that $\lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = \infty$. So if we were to calculate exact epsilons, based on similarity to a function which isn't that great, it wouldn't get that precise, and probably would be messy. To get our hands on something more touchable, we prove $N_2 = 15$. \square

Proof for $N_2 = 15$

This indirect proof will be done by looking at several cases, we always assume $g(n)$ to be odd, as a premise. One can check, that $g(15) = 105 = 3 \cdot 5 \cdot 7$, and $g(16) = 140 = 4 \cdot 5 \cdot 7$. Let $t \in S_n$ be a permutation with order of $g(n)$, and S_t be denoting the set of numbers corresponding to the cycle lengths in t . First we show, that we can always choose such t , that S_t only contains powers of different primes (and a few ones, to fill up the space if needed). Assume we have $k \in S$, where $k = p \cdot q$, which are coprimes, and without loss of generality $p > q$. We can replace k with p and q in S , then its elements have a smaller sum, but the same least common multiple, namely $k = p \cdot q \geq 2p > p + q$, obviously there are corresponding elements in S_n , from now on we will assume t is one of them.

First part: we assume that $\exists p^k \in S$, where p is a prime, and $k > 1$. Notice that there is a 2-power d where $(p-1)p^{k-1} > d > p$, and $d + p^{k-1} < p^k$ while $p^{k-1} \cdot d > p^k$, thus we have a contradiction.

Second part: we only have primes now in S , and we do have the number 11 among them. If we have a 3 also, lets change 11 and 3 to 4 and 9, now we increased the least common multiple, while we didnt increase the sum, If we dont have a 3, change 11 to 3 and 4, or even 3 and 8, and we have the same argument.

Third part: $11 \notin S$, but there are bigger primes, of which, we denote the smallest by p . By Bertrand's postulate (which was proved by Chebyshev in 1852), there is a prime q , for which $p-1 > q > \frac{p-1}{2}$, we take the largest possible q . We know $q \in S$

cannot be, because $p > q \geq 11$, and S has no such elements. Now we can change p to q and 2, and obviously the sum decreases, while the least common multiple increases.

Fourth part: now S has only prime elements, which are all smaller than 11, namely $S \subset \{3, 5, 7\}$. We've seen, that for $n = 16$, we have a bigger $g(n)$, and since $g(n)$ is obviously monotone, that proves, the largest index where $g(n)$ is odd, is indeed $15 = 3 + 5 + 7$. \square

These 2 questions we were driven to last semester, and figured the answer out during this one. There are other things we explored with Zoli, but not yet in a phase to include here.

References

- [1] William Miller, *The Maximum Order of an Element of a Finite Symmetric Group*, American Mathematical Monthly, Volume 94, Issue 6 (Jun. - Jul., 1987), 497-506. .
- [2] <https://oeis.org/A000793>