The Lelek fan as a projective Fraïssé limit

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Introduction

In mathematics, it is often useful to think of interesting structures as the limit, or inverse limit, of smaller structures. In some cases, viewing a space X as a (co)limit of finite structures can allow us to understand the structure on X better, using finite combinatorial properties of the underlying finite sets.

Fraïssé theory and its dual, projective Fraïssé theory, give us a helpful framework to achieve this. In particular, projective Fraïssé theory aids us in understanding topological structures, such as various continua, as inverse limits. During my research thus far, I have gained an understanding of the foundations of projective Fraïssé theory, and possibly the most well-known projective Fraïssé limit, the Pseudo-arc. This semester I looked at another interesting continuum which can be understood as a projective Fraïssé limit, the Lelek fan.

Projective Fraïssé theory

First, we shall quickly recall the main ideas and statements of projective Fraïssé theory. For a fixed first-order language $\mathcal{L}$, we define an $\mathcal{L}$-sructure M to be a topological $\mathcal{L}$-structure if the functions on M are continuous, and the relations on M are closed (as a subset of M^n for various n). Additionally, we also require that $Dom(M)$ be 0-dimensional, second countable, and compact.

Epimorphisms between topological $\mathcal{L}$-structures will be defined as continuous surjective homomorphisms. Usually, we also require that that for an epimorphism $f : X \to Y$, a relation R, and a tuple $\underline{y} \in Y$, if $\underline{y} \in R^Y$, then there must be a tuple $\underline{x} \in X$ for which $\underline{x} \in R^X$.

We define an inverse limit of topological $\mathcal{L}$-structures as the following: for a sequence D_n of topological $\mathcal{L}$-structures, and epimorphisms $\pi_n : D_{n+1} \to D_n$ for all $n \in \omega$, the inverse limit of D_n is defined as the structure D with domain $\{\underline{x} \in \prod D_n \mid \underline{x}(n) = \pi_n(\underline{x}(n+1))\}$, with the topology inherited from $\prod D_n$. That is, the elements of our inverse limit will be the infinite sequences compatible with all epimorphisms π_n.

We define a projective Fraissé class as a class Δ of topological $\mathcal{L}$-structures, where Δ satisfies the Projective Joint Embedding Property (PJEP) and the Projective Amalgamation Property (PAP), which are "reversed" versions of their classical (non-projective) counterparts. We say Δ satisfies PJEP, if for any two structures $A, B \in \Delta$, there exists some $C \in \Delta$ and epimorphisms $\phi_1 : C \to A$ and $\phi_2 : C \to B$. Similarly, Δ satisfies PAP if for any $A, B_1, B_2 \in \Delta$ and epimorphisms $\phi_1 : B_1 \to A$ and $\phi : B_2 \to A$, there exists $C \in \Delta$ and epimorphisms $\rho_1 : C \to B_1$ and $\rho_2 : C \to B_2$ such that $\phi_1(\rho_1(c)) = \phi_2(\rho_2(c))$ for all $c \in C$.

The projective Fraïssé theorem states that if Δ is a projective Fraïssé class, then there is a unique, up to isomorphism, topological $\mathcal{L}$-structure $\mathbb{D}$, which we call the projective Fraïssé limit of Δ, with the following nice properties:

(P1) for every $D \in \Delta$ there is an epimorphism $\phi : \mathbb{D} \to D$.
This property parallels the fact that the *Age* of the Fraïssé limit of a Fraïssé class $\mathcal{K}$ is precisely K.

(P2) if X is a finite discrete topological space and $f : \mathbb{D} \to X$ is continuous, then there exists some $D \in \Delta$, an epimorphism $\phi : \mathbb{D} \to D$ and a function $g : D \to X$ such that $f = g \circ \phi$.
It can be shown that this property is equivalent to the property (P2') below:

(P2') Every open cover $\mathcal{U}$ of $\mathbb{D}$ can be refined by an epimorphism $\phi : \mathbb{D} \to D$ for some $D \in \Delta$. Here we say ϕ "refines" $\mathcal{U}$ if $\{\phi^{-1}(d) \mid d \in D\}$ refines $\mathcal{U}$.

(P3) for a structure $D \in \Delta$ and any two epimorphisms $\phi_1 : \mathbb{D} \to D$ and $\phi_2 : \mathbb{D} \to D$, there exists an automorphism ρ of $\mathbb{D}$ such that $\phi_1 : \phi_2 \circ \rho$.
This property parallels the ultrahomogeneity of the classical Fraïssé limit.

The Lelek-fan

This semester I mainly followed 2 papers from Dana Bartošová and Aleksandra Kwiatkowska: "Lelek fan from a projective Fraïssé limit" and "The universal minimal flow of the automorphism group of the Lelek-fan". The 2 articles discuss how the Lelek-fan can be viewed as a projective Fraïssé limit, prove some remarkable properties of it, and compute the universal minimal flow its homeomorphism group.

We start with a few definitions regarding continua to help define the Lelek-fan. A continuum is defined as a compact connected metric space. A continuum X is called unicoherent if for for any two subcontinua $X_1, X_2 \subseteq X$ for which $X_1 \cup X_2 = X$, the intersection of X_1 and X_2 is connected. A simple example of a non-unicoherent continuum is the circle S^1. A continuum X is hereditarily unicoherent if all subcontinua of X are unicoherent. An arc in a continuum is a homeomorphic image of the closed unit interval $[0, 1]$, and an arc with endpoints a, b will usually be denoted by ab. A dendroid is a hereditarily unicoherent arcwise connected continuum. A ramification point of a dendroid X is a point $x \in X$ for which there exist $a, b, c \in X$ and arcs ax, bx, cx that pairwise intersect only in the point x, i.e. X "branches" in x.

A fan is a dendroid with exactly one ramification point, which we call the "top". A fan X with top t is said to be smooth if whenever we have a convergent sequence of points $x_n \to x$ in X, the arcs tx_n also converge to the arc tx in the Hausdorff metric. The Cantor fan F is the cone over the Cantor set, formally $F = C \times [0, 1]/\sim$, where $(a, b) \sim (c, d)$ if and only if $a = b, c = d$ or $b = d = 0$. An endpoint of a continuum X is a point x for which whenever we have an arc ab in X containing x, we must have $a = x$ or $b = x$. A Lelek fan is defined as a subcontinuum of the Cantor fan with a dense set of endpoints. The following theorem states the uniqueness of the Lelek fan, which means from now on we can talk about "the" Lelek fan.

Theorem. *Any two subcontinua of the Cantor fan with a dense set of endpoints are homeomorphic*

Similarly to the Pseudo-arc, we can show that the Lelek fan is the projective Fraïssé limit of finite graphs, but instead of linear graphs, we will be looking at finite fans. Our first-order language $\mathcal{L}$ consists of a single binary relation R, and our class $\mathcal{F}$ will consist of structures on which R is realized as the directed edge relation of finite fans with edges directed outward from the top. One can prove that $\mathcal{F}$ is indeed a projective Fraïssé class, and in the proof that $\mathcal{F}$ has the projective amalgamation property, the base case once again follows directly from the discrete mountain climbing theorem. the projective Fraïssé limit of $\mathcal{F}$ is the pre-Lelek fan, denoted by $\mathbb{L}$. Similar to the case of the Pseudo-arc, one can arrive at the following theorem.

Theorem. *The class of finite fans $\mathcal{F}$ is a projective Fraïssé class. On the projective Fraïssé limit of $\mathcal{F}$, the relation $R^{\mathbb{L}}$ has only one- and two-element equivalence classes, and the quotient space $\mathbb{L}/R^{\mathbb{L}}$ is the Lelek fan.*

Statement. *Every automorphism of the pre-Lelek fan induces a homeomorphism of the Lelek fan and this assignment respects the quotient map. Because of this, we often write $Aut(\mathbb{L}) \subseteq Homeo(L)$.*

For a smooth fan X we say that a continuous surjection $f : L \to X$ is monotone on segments if $f(v) = w$, where v is the top of L and w is the top of X, and whenever $x, y \in L$ such that $x \in vy$, we have $f(x) \in wf(y)$. Just like in the case of the Pseudo-arc, being the quotient of a projective Fraïssé class comes with a projective universality type of property.

Theorem. *(Projective universality of the Lelek fan)*

1. *Every smooth fan is a continuous image of the Lelek fan by a map that is monotone on segments,*

2. *Suppose X is a smooth fan with a metric d and let f_1, f_2 be continuous surjections from L to X that are monotone on segments. Then, for every $\varepsilon > 0$ there exists a homeomorphism $\phi \in Aut(\mathbb{L})$ of L such that $d(f_1(x), f_2 \circ \phi(x)) < \varepsilon$ for all $x \in L$.*

This theorem immediately implies the following really interesting property as a corollary.

Theorem. *$Aut(\mathbb{L})$ is dense in $Homeo(L)$.*

Using the projective Fraïssé characterization of the Lelek fan, we can prove that the homeomorphism group of the Lelek fan also has the following interesting properties.

Theorem.

1. *$Homeo(L)$ is totally disconnected,*
2. *$Homeo(L)$ has a dense conjugacy class,*
3. *$Homeo(L)$ is simple.*

The next step during my research will be computing the universal minimal flow of the Lelek fan.

Universal minimal flow and extreme amenability

Understanding the various ways a group can act on different spaces can give us insight on the structure of the group, and vice versa. Therefore, it is important to talk about what is, in some sense, the "most complicated" or "most rich" flow of a group, and this is what we call the group's universal minimal flow.

A topological group is a group equipped with a topology for which the group multiplication and inverse-taking functions are continuous. For a topological group G we say that a G-flow is an action of G on some compact Hausdorff space X where the flow is continuous with respect to the topology on G. We will often call X itself the G-flow for ease of notation. We say that a G-flow X is minimal if the orbit of every point is dense in X. The G-flow X is called the universal minimal flow of G if it maps onto every minimal G-flow, that is, for every G-flow Y there is a continuous surjective map from X to Y that respects the G-actions on X and Y.

Theorem. *Every topological group has a universal minimal flow, and this flow is unique up to isomorphism.*

The universal minimal flow of a topological group has close ties with another interesting structural property, extreme amenability. A topological group G is said to be extremely amenable if every G-flow X has a fixed point, that is, there exists some $x \in X$ such that $gx = x$ for all $g \in G$. Using only the definitions given above, one can see the following important statement that connects extreme amenability and the universal minimal flow.

Statement. *A topological group G is extremely amenable if and only if the universal minimal flow of G is a single point.*

This further connects the universal minimal flow to Fraïssé theory, as Fraïssé theory can help us characterize the extreme amenability of a specific class of groups. Namely, the automorphism groups of Fraïssé limits have been shown to be characterized by a combinatorial "Ramsey-like" property of the underlying Fraïssé class. The following theorem is credited to Kechris-Pestov-Todorcevic.

Theorem. *Let $G = Aut(M)$ where M is the Fraïssé limit of the Fraïssé class $\mathcal{K}$. The following are equivalent.*

1. *G is extremely amenable,*
2. *$\mathcal{K}$ is a locally finite Fraïssé order class that has the Ramsey property. (Here, the Ramsey property is a generalization of the finite combinatorial Ramsey coloring theorem to Fraïssé classes.)*

Uniformities and proximity spaces

While reading the article of Bartošová and Kwiatkowska, one will come accross the notion of a uniformity, which was unfamiliar to me initially. To help understand the article better, I chose to take a detour and delve into this general topological concept. In this section, I will talk about the fundamentals of uniformities and proximity spaces, loosely following the relevant sections from the books: R. Engelking - General topology, and Császár Ákos - General topology.

A proximity space $(X, \mathcal{P})$ is a set X equipped with a binary relation $\mathcal{P}$ on subsets of X where $\mathcal{P}$ satisfies the following properties.
For all $A, B, C \subseteq X$, we have:

1. $A\mathcal{P}B$ implies $B\mathcal{P}A$
2. $A \cap B \neq \emptyset$ implies $A\mathcal{P}B$
3. $A\mathcal{P}\mathcal{B}$ and $A \subseteq A'$ and $B \subseteq B'$ implies $A'\mathcal{P}B$
4. $\emptyset\overline{\mathcal{P}}X$ (where $\overline{\mathcal{P}}$ is the negation of $\mathcal{P}$)
5. $A\overline{\mathcal{P}}B$ and $B\overline{\mathcal{P}}C$ implies $A \cup B\overline{\mathcal{P}}C$
6. Whenever $A\overline{\mathcal{P}}B$, there exist $U, V \subseteq X$ such that $U \cap V = \emptyset$ and $A\overline{\mathcal{P}}U^C$ and $B\overline{\mathcal{P}}V^C$ (where $U^C = X - U$ is the complement of U)

These axioms define a relation that tells us which subsets of X are "near" in some sense. For example, in a pseudometric space we can say that sets that are "near" are exactly the sets of distance 0 to each other. Therefore, if $A\mathcal{P}B$ then we say that A is near B, and if $A\overline{\mathcal{P}}B$ then we say that A and B are apart.

For a set E we denote the diagonal of $E \times E$ as Δ:

$$\Delta = \{(x, x) \mid x \in E\} \subseteq E \times E$$

An entourage of the set E is a set $D \subseteq E \times E$ that contains Δ and for which $D = -D = \{(y, x) \mid (x, y) \in D\}$. The set of entourages of a set E will be denoted by $\mathcal{D}_E$. For $A, B \in \mathcal{D}_E$ we write

$$A \circ B = \{(x, z) \mid \exists y \in E \text{ such that } (x, y) \in A \text{ and } (y, z) \in B\}$$

A uniformity on a set E is a set $\mathcal{U} \subseteq \mathcal{D}_E$ that satisfies the following properties:

1. $V \in \mathcal{U}$ and $V \subseteq W \in \mathcal{D}_E$ implies $W \in \mathcal{U}$
2. $V, W \in \mathcal{U}$ implies $V \cap W \in \mathcal{U}$
3. For every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ for which $V \circ V \subseteq U$
4. $\bigcap \mathcal{U} = \Delta$

A set E equipped with a uniformity $\mathcal{U}$ is called a uniform space. The entourages of E can be understood as various concepts of "closeness", where $x \in E$ and $y \in E$ are U-close for some $U \in \mathcal{U}$ if $(x, y) \in U$.

Similarly, we can define subsets $A, B \subseteq E$ to be U-close if there are $a \in A$ and $b \in B$ such that a and b are U-close. This gives us a way to see a uniform space as a proximity space where our proximity relation is

$$A \mathcal{P}_{\mathcal{U}} B \text{ if and only if } A \text{ and } B \text{ are } U\text{-close for every } U \in \mathcal{U}$$

Furthermore, we can view proximity spaces (and therefome uniform spaces) as topological spaces by letting the neighborhood basis of each element x consist of precisely the sets that are $\mathcal{P}$-apart from x. In other words, one can check that for a proximity space $(E, \mathcal{P})$, the following neighborhood bases give a toplogy on E:
For every $x \in E$

$$\tau_{\mathcal{P}_x} = \{U \subseteq E \mid \{x\} \overline{\mathcal{P}} U\}$$

Using elementary general topological ideas, we can arrive at the following theorem of base importance about uniform spaces. (A few phrases in the theorem will not be defined, as it would be going into too much detail for this short summary.)

Theorem. *(Completion of a uniform space)*
Let $(E,\mathcal{U})$ be a uniform space, let $E \subseteq E'$ and suppose we have a bijection ν between $E' - E$ and the non-convergent round Cauchy filters of $(E,\mathcal{U})$. Furthermore, suppose ν satisfies that for every $x \in E$ we have $\nu(x) = \tau_{\mathcal{U}}(x)$ (where $\tau(x)$ is the topological neighborhood basis of x). Then there exists a uniformity $\mathcal{U}'$ on E' for which $(E',\mathcal{U}')$ is complete and a reduced extension of $(E,\mathcal{U})$ and for every $x \in E'$ we have that the trace of the $\tau_{\mathcal{U}'}$-neighborhood basis of x is $\tau_{\mathcal{U}'}(x) \cap E = \nu(x)$. Furthermore, if $(E,\mathcal{U})$ was precompact then $(E',\mathcal{U}')$ is compact.