The Lelek fan as a projective Fraïssé limit

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Motivation

- We like to think of structures as (inverse) limits of smaller structures.
- Fraïssé theory and projective Fraïssé theory helps us:
  - → Construct universal structures as (co)limits of finite structures.
  - → Analyze these limits using combinatorial properties of the underlying finite structures.
- Projective Fraïssé theory is useful for talking about:
  - Topological structures,
  - Various continua, such as:
    - The Pseudo-arc,
    - **The Lelek fan**,
    - many others...

Projective Fraïssé theory

Topological $\mathcal{L}$-structure

- First-order structure for a language $\mathcal{L}$.
- Functions are continuous, relations are closed.
- compact, second countable (M_2), zero-dimensional.

Epimorphism

An *epimorphism* is a continuous surjective homomorphism between topological $\mathcal{L}$-structures. There is usually another requirement, but in our case this will suffice.

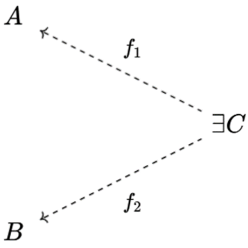
Inverse limits arise as infinite sequences compatible with epimorphisms.

$$D_0 \xleftarrow{\phi_0} D_1 \xleftarrow{\phi_1} D_2 \xleftarrow{\phi_2} D_3 \xleftarrow{\phi_3} D_4 \xleftarrow{\phi_4} \dots$$

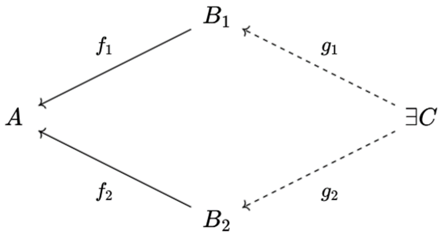
Projective Fraïssé theory

Projective Fraïssé class

A class of finite topological $\mathcal{L}$-structures (for some fixed first-order language $\mathcal{L}$), for which the following properties hold:



Projective Joint Embedding Property (PJEP)



Projective Amalgamation Property (PAP)

Projective Fraïssé Theory

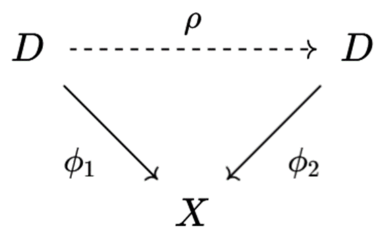
The projective Fraïssé theorem

For a projective Fraïssé class Δ, there exists a unique (up to isomorphism) topological $\mathcal{L}$-structure D such that:

P1) There is an epimorphism from D to any $X \in \Delta$ (Projective universality),

P2) Every open cover of D can be refined by an epimorphism $D \to X$ for some $X \in \Delta$,

P3) The following diagram can be made commutative for any $X \in \Delta$ by a suitable automorphism ρ of D: (Projective ultrahomogeneity)



Continua and dendroids

Continuum

A *continuum* is a compact connected metric space.

Hereditarily unicoherent

A continuum X is called *hereditarily unicoherent* if for any two subcontinua $X_1, X_2 \subseteq X$ their intersection $X_1 \cap X_2$ is connected.

A simple example of a non-unicoherent continuum is the circle S^1.

Arc

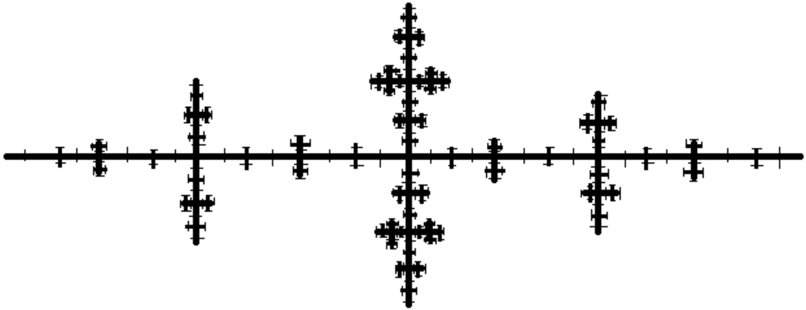
An *arc* in a continuum is a homeomorphic image of the closed unit interval $[0, 1]$. An arc with endpoints $a, b \in X$ will be denoted by ab.

Dendroid

A *dendroid* is an arcwise connected hereditarily unicoherent continuum.

Continua and dendroids

Dendroids are kind of like "limits of trees":



Ramification point

A *ramification point* of a dendroid is a point x for which there exist arcs ax, bx, cx that pairwise intersect only in the point x.

Ramification points are the "branching points".

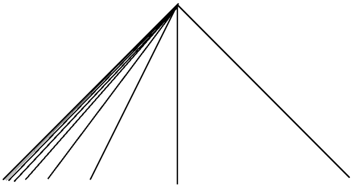
Fans

Fan

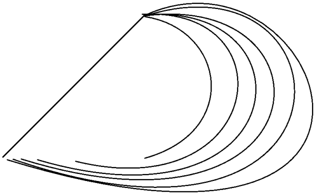
A *fan* is a dendroid with exactly one ramification point. This point is called the "top" of the fan.

Smooth fan

A fan with top t is called *smooth* if whenever we have a convergent sequence $x_n \to x$, the corresponding arcs also converge $tx_n \to tx$ in the Hausdorff metric.



A smooth fan



A non-smooth fan

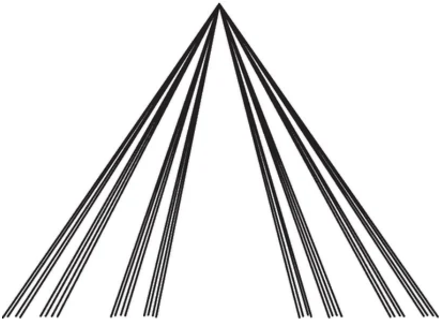
The Cantor fan and Lelek fans

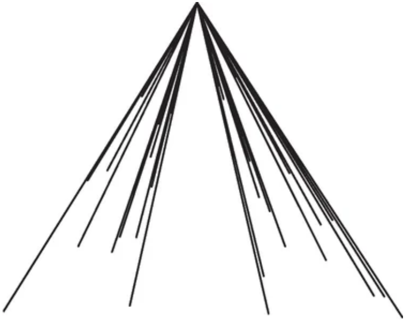
The Cantor fan

The *Cantor fan* is defined as the cone over the Cantor set. (left)

Lelek fan

A *Lelek fan* is defined as a subfan of the Cantor fan with a dense set of endpoints. (right)





The Lelek fan

Theorem

Any two subcontinua of the Cantor fan with a dense set of endpoints are homeomorphic.

Therefore Lelek fans are unique up to homeomorphism, so we can talk about *The* Lelek fan. Now we will look at the Lelek fan from a projective Fraïssé viewpoint.

Theorem

The class $\mathcal{F}$ of finite fans is a projective Fraïssé class.

Let us denote the projective Fraïssé limit of $\mathcal{F}$ by $\mathbb{L}$.

Theorem

The relation $R^{\mathbb{L}}$ is an equivalence relation with equivalence classes of size 1 or 2, and the quotient space $\mathbb{L}/R^{\mathbb{L}}$ is homeomorphic to the Lelek fan.

Properties of the Lelek fan

Being the quotient of a projective Fraïssé limit comes with a type of projective universality property.

Projective universality of the Lelek fan

1. Every smooth fan is a continuous image of the Lelek fan by a map that is monotone on segments,
2. Suppose X is a smooth fan with a metric d and let f_1, f_2 be continuous surjections from L to X that are monotone on segments. Then, for every $\varepsilon > 0$ there exists a homeomorphism $\phi \in Aut(\mathbb{L})$ of L such that $d(f_1(x), f_2 \circ \phi(x)) < \varepsilon$ for all $x \in L$.

The previous theorem immediately implies the following really interesting property as a corollary.

Theorem

$Aut(\mathbb{L})$ is dense in $Homeo(L)$.

Properties of the Lelek fan

Using the projective Fraïssé characterization of the Lelek fan, we can prove that the homeomorphism group of the Lelek fan also has the following interesting properties.

Theorem

1. *$Homeo(L)$ is totally disconnected,*
2. *$Homeo(L)$ has a dense conjugacy class,*
3. *$Homeo(L)$ is simple.*

The next step will be to compute the universal minimal flow of the homeomorphism group of the Lelek fan.

Universal minimal flows

It is helpful to understand all the ways a group can act on various spaces. In some sense, the universal minimal flow is the "most rich" or "most complicated" action a group can have on any space.

Topological group

A *topological group* is a group G equipped with a topology for which multiplication and inverse-taking are continuous.

Flow

A *flow* of a topological group G is an continuous action of G on some compact Hausdorff space X.

Minimal flow

We say that a G-flow X is *minimal* if the orbit of every point is dense in X.

Universal minimal flows

Universal minimal flow

A minimal G-flow X is called the *universal minimal flow* of G if it maps onto every minimal G-flow, that is, for every G-flow Y there is a continuous surjective map from X to Y that respects the G-actions on X and Y.

Theorem

Every topological group has a universal minimal flow, and this flow is unique up to isomorphism.

A special case of the universal minimal flow: *extreme amenability*. This property has close ties with Fraïssé classes and projective Fraïssé classes.

Universal minimal flows

Extreme amenability

A topological group G is called *extremely amenable* if every G-flow X has a fixed point, i.e. there exists some $x \in X$ such that $gx = x$ for all $g \in G$.

Statement

A topological group G is extremely amenable if and only if the universal minimal flow of G is a single point.

Theorem of Kechris, Pestov, Todorcevic

Let $G = Aut(M)$ where M is the Fraïssé limit of the Fraïssé class $\mathcal{K}$. The following are equivalent.

1. G is extremely amenable,
2. $\mathcal{K}$ is a locally finite Fraïssé order class that has the Ramsey property. (Here, the Ramsey property is a generalization of the finite combinatorial Ramsey coloring theorem to Fraïssé classes.)

Proximity spaces and Uniformities

These are general topological concepts used in the construction of the Universal minimal flow of $Homeo(L)$.

Proximity space

A *proximity space* is a set X equipped with a binary relation $\mathcal{P}$ on subsets of X where $\mathcal{P}$ satisfies the following properties.
For all $A, B, C \subseteq X$, we have:

1. $A\mathcal{P}B$ implies $B\mathcal{P}A$
2. $A \cap B \neq \emptyset$ implies $A\mathcal{P}B$
3. $A\mathcal{P}\mathcal{B}$ and $A \subseteq A'$ and $B \subseteq B'$ implies $A'\mathcal{P}B$
4. $\emptyset\overline{\mathcal{P}}X$ (where $\overline{\mathcal{P}}$ is the negation of $\mathcal{P}$)
5. $A\overline{\mathcal{P}}C$ and $B\overline{\mathcal{P}}C$ implies $(A \cup B)\overline{\mathcal{P}}C$
6. Whenever $A\overline{\mathcal{P}}B$, there exist $U, V \subseteq X$ such that $U \cap V = \emptyset$ and $A\overline{\mathcal{P}}U^C$ and $B\overline{\mathcal{P}}V^C$ (where $U^C = X - U$ is the complement of U)

In pseudometric spaces, sets that are "near" $=$ sets that have distance 0.

Proximity spaces and Uniformities

For a set E we denote the *diagonal* of $E \times E$ as Δ:

$$\Delta = \{(x, x) \mid x \in E\} \subseteq E \times E$$

An *entourage* of the set E is a set $D \subseteq E \times E$ that contains Δ and for which $D = -D = \{(y, x) \mid (x, y) \in D\}$. The set of entourages of a set E will be denoted by $\mathcal{D}_E$. For $A, B \in \mathcal{D}_E$ we write

$$A \circ B = \{(x, z) \mid \exists y \in E \text{ such that } (x, y) \in A \text{ and } (y, z) \in B\}$$

A *uniformity* on a set E is a set $\mathcal{U} \subseteq \mathcal{D}_E$ that satisfies the following properties:

1. $V \in \mathcal{U}$ and $V \subseteq W \in \mathcal{D}_E$ implies $W \in \mathcal{U}$
2. $V, W \in \mathcal{U}$ implies $V \cap W \in \mathcal{U}$
3. For every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ for which $V \circ V \subseteq U$
4. $\bigcap \mathcal{U} = \Delta$

Proximity spaces and Uniformities

A set E equipped with a uniformity $\mathcal{U}$ is called a *uniform space*. We say $x \in E$ and $y \in E$ are U-close for some $U \in \mathcal{U}$ if $(x, y) \in U$. Similarly, we say subsets $A, B \subseteq E$ to be U-close if there are $a \in A$ and $b \in B$ such that a and b are U-close.

This gives us a way to see a uniform space as a proximity space where our proximity relation is:

$$A \mathcal{P}_{\mathcal{U}} B \text{ if and only if } A \text{ and } B \text{ are } U\text{-close for every } U \in \mathcal{U}$$

We can view proximity spaces (and therefome uniform spaces) as topological spaces by considering the topology given by the following neighborhood bases: For every $x \in E$

$$\tau_{\mathcal{P}_x} = \{U \subseteq E \mid \{x\} \overline{\mathcal{P}} U\}$$

Proximity spaces and Uniformities

The following theorem is of base importance about uniform spaces. (A few phrases in the theorem will not be defined, as it would be going into too much detail for this short presentation.)

Completion of a uniform space

Let $(E,\mathcal{U})$ be a uniform space, let $E \subseteq E'$ and suppose we have a bijection ν between $E' - E$ and the non-convergent round Cauchy filters of $(E,\mathcal{U})$. Furthermore, suppose ν satisfies that for every $x \in E$ we have $\nu(x) = \tau_{\mathcal{U}}(x)$ (where $\tau(x)$ is the topological neighborhood basis of x). Then there exists a uniformity $\mathcal{U}'$ on E' for which $(E',\mathcal{U}')$ is complete and a reduced extension of $(E,\mathcal{U})$ and for every $x \in E'$ we have that the trace of the $\tau_{\mathcal{U}'}$-neighborhood basis of x is $\tau_{\mathcal{U}'}(x) \cap E = \nu(x)$. Furthermore, if $(E,\mathcal{U})$ was precompact then $(E',\mathcal{U}')$ is compact.

Thank you for your attention!

Source of the Cantor fan Lelek fan image:
https://www.researchgate.net/figure/The-Cantor-fan-on-the-left-and-the-Lelek-fan-on-the-right_fig1_383566343