Surreal numbers

In this $5 = \{0, 1, 2, 3, 4 \mid \ \}$ page we shall have a glimpse of surreal numbers.

Our Viewpoint

The direction from which we look at it is what is beyond $\mathbb{R}$, in the sense that we see "more" in $\mathbb{R}$ than in $\mathbb{Q}$. Can't we go further? Of course, we can go to $\mathbb{C}$, but we lose our order, so is there some bigger ordered field, from which we may see something "more" and how can we know for sure that we can stop then, because there isn't anything bigger, that has enough beauty for our investigation? Fortunately there is a beautiful answer to our request, the surreal numbers (with the premise that the reader finds it beautiful enough). Out of respect for Conway, we just use the term number.

Intuition

In kindergarten we built the real numbers from rationals with Dedekind cuts and the ordinals with von Neumann's help. We might not have recognized that the two are special cases of a more general construction, in Dedekind cuts we forbid empty side while in von Neumann ordinals we "forbid" non-empty "right" side (we didn't even speak about right side). Let $x = \{L \mid R\}$ a number that is strictly larger than any element of L and strictly smaller than any element of R, numbers form an ordered Field **No** (i.e. a field whose domain is a proper class), which is the universal ordered field and it contains the ordinals as well. We can further generalize the definition of numbers and get the definition of games at the price that our total order expands into a partial order and we lose some of the field properties. We will construct games and numbers (i.e. specific games) in ZFC using transfinite induction in every definition and proof.

Formal definitions

For an ordinal α, let $\overline{M}_\alpha := \{(L, R) : L, R \subseteq \bigcup_{\beta<\alpha} \overline{M}_\beta\}$ be the games born on or before day α, $\overline{N}_\alpha := \overline{M}_\alpha \setminus \bigcup_{\beta<\alpha} \overline{M}_\beta$ games born on day α. For the sake of intuition about the order on games and to follow Conway's notation, we will write $\{L \mid R\}$ instead of (L, R) or use the notion $\{x^L \mid x^R\}$ where x^L, x^R means a typical member of L, R.

x is a game $:\Leftrightarrow x \in \overline{M}_\alpha$ for some α.

For x, y games, $x \le y$ iff $\nexists x^L : x^L \ge y \wedge \nexists y^R : x \ge y^R \quad (=:\Leftrightarrow \le \wedge \ge; \quad <:\Leftrightarrow \le \wedge \not\ge)$

We use $\equiv$ for identity between games, while $=$ for the defined equality.

x is a number $:\Leftrightarrow$ x is a game and $\nexists x^L \ge x^R$ and all members are numbers. (We say two number is the same if they $=$, of course we can't take the whole equivalence class as it's proper Class, but for example we can take the equivalence class in the first $\overline{N}_\alpha$)

We can define the operations on games and some of the field properties will work even with $\equiv$.

(In some cases we discuss why it "should" be the definition.)

$x + y :\equiv \{x^L + y, x + y^L \mid x^R + y, x + y^R\} \quad (x^L < x \Rightarrow x^L + y < x + y)$

$0 :\equiv \{\ \mid\ \} \equiv \{\emptyset \mid \emptyset\}$

$-x :\equiv \{-x^R \mid -x^L\} \quad (x^R > x \Rightarrow -x^R < -x)$

$x \cdot y :\equiv \{x^L \cdot y + x \cdot y^L - x^L \cdot y^L, x^R \cdot y + x \cdot y^R - x^R \cdot y^R \mid x^L \cdot y + x \cdot y^R - x^L \cdot y^R, x^R \cdot y + x \cdot y^L - x^R \cdot y^L\}$

$(x^L < x \wedge y^L < y \Leftrightarrow 0 < x - x^L \wedge 0 < y - y^L \Rightarrow 0 < (x - x^L) \cdot (y - y^L) \Leftrightarrow x^L \cdot y + x \cdot y^L - x^L \cdot y^L < x \cdot y)$

$1 :\equiv \{0 \mid\ \}$

Because of the complexity of division we shall just give the formula for y, where $x \cdot y = 1$, where $x \equiv \{0, x^L \mid x^R\} > 0$, and we only use positive x^L:

$y = \{0, \frac{1+(x^R-x)\cdot y^L}{x^R}, \frac{1+(x^L-x)\cdot y^R}{x^L} \mid \frac{1+(x^L-x)\cdot y^L}{x^L}, \frac{1+(x^R-x)\cdot y^R}{x^R}\}$

where we can find y^L, y^R in the definition, by that we mean the previous options for y, for example in the beginning we only "know" that y has a 0 left option, then we can use that to generate other left and right options and so on.

Remark: Somebody can have the idea that we can define addition as $x + y :\equiv \{x^L + y^L \mid x^R + y^R\}$, but unfortunately this is not "enough", because in that case for example $\{\mathbb{N} \mid\ \} + \{\mathbb{N} \mid\ \} = \{\mathbb{N} \mid\ \}$ and we are in a Field (at least we want to be in a Field), therefore $\{\mathbb{N} \mid\ \} = 0$, but that is a contradiction.

About the induction

In the inductive proofs, we often use this to prove the statement for x, y with born days α, β that for x^L, y with born days δ, β $(\delta < \alpha)$ and for similar cases the statement is true. (We don't need any first step in the induction, since the definition) Let us see an exception: For x, y, z games $x \le y \wedge y \le z \Rightarrow x \le z$

Let the born days be $\alpha, \beta, \gamma \quad \lightning\ x \not\le z$ then we have two cases:

First: $\exists x^L \ge z$, with the same statement for β, γ, δ we have $x^L \ge y \ \lightning_1$ $(\delta < \alpha)$

Second: $\exists z^R : x \geq z^R$, with the same statement for δ, α, β we have $y \geq z^R \lightning_2$ $(\delta < \gamma)$

The problem that can arise in somebody's mind is that we have not only reduced an ordinal, but permuted them as well, fortunately if we define an order on finite tuples of ordinals in this way $((\alpha_0, \dots, \alpha_{n-1}) > (\alpha_{\pi(0)}, \dots, \alpha_{\pi(i-1)}, \beta, \alpha_{\pi(i+1)}, \dots, \alpha_{\pi(n-1)})$ where $\beta < \alpha_{\pi(i)}$ and π is a permutation) we get a well partial order (that can even be extended to a well-order), so we can do a nice induction on it.

About the identity

As we said earlier there are (quite a lot of) cases where we not only can prove $=$ but $\equiv$ as well for example: $x + y \equiv y + x$ can be used in $x \cdot y \equiv y \cdot x$ as: $x \cdot y \equiv \{x^L \cdot y + x \cdot y^L - x^L \cdot y^L, x^R \cdot y + x \cdot y^R - x^R \cdot y^R \mid x^L \cdot y + x \cdot y^R - x^L \cdot y^R, x^R \cdot y + x \cdot y^L - x^R \cdot y^L\} \equiv \{y^L \cdot x + y \cdot x^L - y^L \cdot x^L, y^R \cdot x + y \cdot x^R - y^R \cdot x^R \mid y^L \cdot x + y \cdot x^R - y^L \cdot x^R, y^R \cdot x + y \cdot x^L - y^R \cdot x^L\} \equiv y \cdot x$. (This can be regarded as a sign that our definitions are quite good.)

Reals and Ordinals in No

It's quite interesting how the rationals are built in this setup in **No** , one can check that $\{0 \mid 1\} = \frac{1}{2}$ and surprisingly $\{0 \mid \frac{1}{2}\} = \frac{1}{4}$. On all finite days all the dyadic rationals are born and then on day ω reals were born as well as some other numbers around the dyadic rationals and the two extreme numbers. Ordinals can also be embedded, let $\overline{\alpha} := \{\overline{\beta} : \beta < \alpha \mid \}$, but this only preserves the order, it can clearly be seen by remembering that **No** is a Field that, for example, has commutative addition so $1 + \overline{\omega} = \overline{\omega} + 1$ and $1 + \omega = \omega \neq \omega + 1$, but one can check that $\overline{\alpha + 1} = \overline{\alpha} + 1$, so the "successor" ordinals are the same. (For those who always wanted to go one step left from a limit ordinal, now you can, because we can subtract 1 from anything in **No**.)

Motto

One could already notice that our definitions satisfy the motto: "Whatever is not forbidden, is permitted.", to "prove" this we have the simplicity theorem:

If x is a game and y is a number, and $x^L \ngeq y \ngeq x^R$ and no y^L or y^R satisfies that, then $x = y$.

Sign-expansions

Similarly as we did with the games we can define M_α to be numbers formed by $\overline{M}_\alpha$ and $N_\alpha := M_\alpha \setminus \bigcup_{\beta<\alpha} M_\beta$. (Note that N_α usally not equal to numbers formed by $\overline{N}_\alpha$, as they can appear earlier as an other game.)

We can define the βth approximation of x for $\beta \le \alpha$, where $x \in N_\alpha$, by looking at "where are" x in $\bigcup_{\gamma<\beta} M_\gamma$ and take that number from N_β, that are on the "same" spot.

$x_\beta := \{\{y \in \bigcup_{\gamma<\beta} M_\gamma : y < x\} \mid \{y \in \bigcup_{\gamma<\beta} M_\gamma : x < y\}\}$

And $x_\beta := x$ if $\beta > \alpha$ (Note: $x_\alpha = x$)

We can define s_β as the sign of $x - x_\beta$, where we use $+, -, 0$ as values, that s is the sign-expansion of x, it's clear that s has no 0 value for $\beta < \alpha$ and it's 0 beyond.

We can order such sign-sequences lexicographically and it's clear that it's the same order as that on numbers, as we can think of sign-expansions as a map to the number from 0 on the tree of numbers which tells us wich dirrection should we go (or stay there forever).

We can find a number corresponding to an arbitrary sign-sequence (no 0 value for $\beta < \alpha$ and it's 0 beyond), by induction we know the approximation and then $x := \{\{x_\beta : s_\beta = -\} \mid \{x_\beta : s_\beta = +\}\}$

The ω-map

Let us say $x \sim y$ (commensurate) iff $\exists n \in \mathbb{N} \subseteq \mathbf{No} : x < n \cdot y \;\wedge\; n \cdot x > y$, where x, y positive numbers. Note that it's a convex equivalence relation, and we could switch n to $r \in \mathbb{R}$, therefore each equivalence class have a unique simplest element (leader).

Let us build the ω-map on these leaders as $\omega^0 := 1$ and then we take the simplest in an orderly manner: $\omega^x := \{0, r \cdot \omega^{x^L} \mid r \cdot \omega^{x^R}\}$

We mapped all leaders, and $\omega^0 = 1,\ \omega^{-x} = \frac{1}{\omega^x},\ \omega^{x+y} = \omega^x \cdot \omega^y$.

Then with the properties of the reals:

$x = r_0 \cdot \omega^{y_0} + x_1$ (for arbitrary number x exists unique $r_0 \in \mathbb{R}$ and y_0, x_0 numbers)

Than we know that $|r \cdot x_1| < x$ (for arbitrary $r \in \mathbb{R}$)

Of course we can continue (usually it doesn't end in finite or in ω steps).

We can define inductively the normal forms, suppose that for every $\beta < \alpha$ the β-term $(r_\beta \cdot \omega^{y_\beta})$ of every x number is already defined, then let $\sum_{\beta<\alpha} r_\beta \cdot \omega^{y_\beta}$ be the simplest number, with these terms for $\beta < \alpha$. From the induction we already know the terms of x $(\beta < \alpha)$, so we can write $x = \sum_{\beta<\alpha} r_\beta \cdot \omega^{y_\beta} + x_\alpha$, by then if $x_\alpha = 0$, then let the α-term be 0, else $x_\alpha = r_\alpha \cdot \omega^{y_\alpha} + z$, like before and let the α-term be $r_\alpha \cdot \omega^{y_\alpha}$.

From the definition we see, that the partial sums of $x \in N_\gamma$ is always from M_γ, so it can't be distinct for all ordinals, so than from some ordinals the terms must be 0.

Some Algebra

Let $Y = (y_\beta : \beta < \alpha)$ descending series, and $r_y \in \mathbb{R}$ if $y \in Y$, otherwise 0, than with the previous definition we can define $\sum_{y \in \mathbf{No}} r_y \cdot \omega^y := \sum_{\beta<\alpha} r_\beta \cdot \omega^{y_\beta}$.

Then $\sum_{y\in\mathbf{No}} r_y \cdot \omega^y + \sum_{y\in\mathbf{No}} s_y \cdot \omega^y = \sum_{y\in\mathbf{No}} (r_y + s_y) \cdot \omega^y$.

Now we can write x in normal form $\sum_{y\in\mathbf{No}} r_y \cdot \omega^y$.

All positive x, has a unique n-th root: $x = r \cdot \omega^y \cdot (1+\delta)$, where δ has an absolute value less than all real number (infinitesimal), then $r^{\frac{1}{n}} \cdot \omega^{\frac{y}{n}} \cdot [1 + \frac{1}{n} \cdot \delta + \frac{1}{n} \cdot (\frac{1}{n} - 1) \cdot \delta^2 + \dots]$ is good.

Moreover every odd degree polinomial with coefficients from **No** has a root in **No**.

From the real-closure of **No** we can prove that **No** is a universally embedding totally ordered Field, such that every subfield K (set) and ordered field F (set) $K \subseteq F$, then there exists $F' \subseteq \mathbf{No}$ field isomorph to F with identity on K.

And because of the same size of proper Classes, **No** is the unique universally embedding totally ordered Field up to isomorphism.