New results about Q and Δ-spaces

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May 2026

This semester, I continued the research I had started in the previous semester on Q-spaces and Δ-spaces. In the following, I present the results (without proofs), that Ákos Székely and I obtained. Our main result Theorem 2.1 provides complete to some and partial answers to other problems appeared recently in the literature of these spaces.

We also prove a new result concerning Lindelöf Q-spaces Theorem 3.2. This yields a number of nonexistence results for large Lindelöf, locally compact, compact, and countably compact Q-spaces.

We are currently writing up the paper, and these results will be published soon.

1 Introduction

First, recall the definition of Q-spaces.

Definition 1.1. A topological space X is called a Q-space if every subset of X is a G_δ-set.

Originally, Q-spaces were considered, as uncountable subset of the real line, such subsets are usually called nowadays *Q-sets*. The existence of a Q-set clearly implies $2^{\omega_1} = 2^\omega$, so, for example, under CH, no Q-set exists. On the other hand, it is known that under Martin's axiom (MA) and the negation of CH, every uncountable subset of of the real line with cardinality less than $\mathfrak{c}$ is a Q-set [13].

Later, Zoltán Balogh [1] constructed a T_3 non-σ-discrete Q-space in ZFC, and by further refining his method, he even gave an example of a T_4 paracompact Q-space [2].

Now recall the definition of Δ-spaces.

Definition 1.2. A topological space X is a Δ-space if for every decreasing sequence of subsets $\{D_n : n \in \omega\}$ of X with empty intersection, there exists a decreasing sequence of open subsets $\{U_n : n \in \omega\}$ of X such that for every $n \in \omega : U_n \supseteq D_n$ and

$$\bigcap_{n\in\omega} U_n = \emptyset.$$

It is easy to see that every Q-space is also a Δ-space. On the other hand, if X is a non-σ-discrete Q-space, then its *Alexandroff duplicate* is a Δ-space, but not a Q-space.

The notion of a Δ-*set* can be defined analogously to that of a Q-set: namely, it is an uncountable subset of the real line that is a Δ-space. Furthermore, it follows from

[7, Corollary 4.2] that if $o(X)^\omega \leq |X|$ and $\mathrm{cf}(o(X)) > \omega$, then X is not a Δ-space. Consequently, every Δ-set has cardinality $< \mathfrak{c}$. Therefore under MA, every Δ-set is also a Q-set. We note that under MA, every subset of the real line of cardinality $< \mathfrak{c}$ is meagre. The following problem, posed in [8] and [11]: *Does there exist a crowded Baire*[1] *Δ-space?* In Section 2, we show that the existence of such spaces is equiconsistent with the existence of a measurable cardinal.

We also mention that Δ-spaces have recently become a topic of interest in the study of C_p-spaces, since it was shown in [9] that a Tychonov space X is a Δ-space if and only if the locally convex space $C_p(X)$ is *distinguished*.

2 On measure and category for Q and Δ-spaces

In this section, we answer several problems that have appeared in the literature. [11, Problem 6.10] asks the following: *Is it true that if X admits a strictly positive σ-additive measure vanishing on points, then X is not a Δ-space?* Recall that a measure is strictly positive if and only if it assigns positive measure to every nonempty open set.

The answer is negative in general. Indeed, if there exists a Δ-space X such that every nonempty open set is uncountable (that is, $\Delta(X) \geq \omega_1$), then one can define a measure μ by assigning ∞ to uncountable sets and 0 to countable ones. In this case, μ is a strictly positive σ-additive measure vanishing on points on the Δ-space X. Moreover, there exists a Δ-space X, in fact even a Q-space, for which $\Delta(X) \geq \omega_1$. For example, consider the following space: take ω_1 many pairwise disjoint copies of $\mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers with the usual topology, and let $\mathcal{I}$ be the ideal on it consisting of the nowhere dense sets. Endow it with the following topology:

$$\tau := \{\emptyset\} \cup \{U \subseteq \bigsqcup_{i\in\omega_1} \mathbb{Q} \ : \ \bigsqcup_{i\in\omega_1} \mathbb{Q} \setminus U \in \mathcal{I}\}.$$

It is easy to see that the resulting space X is a Q-space with $\Delta(X) = \omega_1$.

We note that the above space is only T_1. However, for example, under Martin's axiom, there are Q-sets with $\Delta(X) \geq \omega_1$.

The preceding examples can be regarded as trivial from the measure-theoretic point of view, but one can instead ask the following: Is it true that if X admits a strictly positive σ-additive *probability* measure, vanishing on points, then X is not a Δ-space? In what follows, we answer this question and also provide a complete solution to [8, Problem 4.1] and to the first part of [11, Problem 6.9]: *Does every Baire Δ-space have an isolated point?* We also give a partial answer to the second part of [11, Problem 6.9]: *Are there uncountable (non σ-discrete) Baire Q-spaces?*

Theorem 2.1. *The following are equiconsistent:*

(1) There exists a measurable cardinal;

(2) There exists a crowded Baire T_1 Δ-space;

(3) There exists a crowded Baire T_4 Q-space[2]*;*

[1] A topological space is Baire, if every nonempty open set is not meagre.

[2] We note that a crowded T_1 Baire space cannot be σ-discrete.

(4) There exists a T_1 Δ-space which admits a strictly positive probability measure vanishing on points;

(5) There exists a T_3 Q-space which admits strictly positive probability measure vanishing on points.

The proof of the theorem above is outlined as follows. The implication $Con(2) \implies Con(1)$ was proved in [7, Corollary 3.2]: if there exists a crowded Baire T_1 Δ-space, then there is an inner model containing a measurable cardinal. Moreover, (3) $\implies$ (2) and (5) $\implies$ (4) are trivial.

For $Con(1) \implies Con(3)$, we assume that κ is a measurable cardinal, and force by adding κ many Cohen reals to the ground model V. Then, in $V[G]$, using a precipitous ideal on κ as the underlying set, we can define a crowded Baire T_4 Q-space.

For $Con(1) \implies Con(5)$, we proceed in a similar way, except that this time we add κ many random reals.

Finally, for $Con(4) \implies Con(1)$, we prove the following measure-theoretic theorem, which is interesting in its own right and from which the implication follows easily.

Theorem 2.2. *Assume that there is no real-valued measurable cardinal, and let μ be a σ-finite (and not identically zero) measure on the space X that vanishes on points. Then there exist infinitely many pairwise disjoint subsets of X of full outer measure.*

3 On the cardinality of Lindelöf Q spaces

In this section, we study the question of whether a non σ-discrete Q-space can be Lindelöf. As we have already seen, under MA and the negation of CH there exists a Q-set $X \subseteq \mathbb{R}$, and since the real line is hereditarily Lindelöf, the Q-space X is Lindelöf. The following natural question is due to Balogh:

Problem 3.1. *Does there exist a Lindelöf Q-space in ZFC?*

It is easy to see that every Lindelöf Q-space is also an L-space (that is, a nonseparable hereditarily Lindelöf space). The only L-space known in ZFC is the space $\mathcal{L}$ constructed by Moore [14]. However, in a recent result, Memarpanahi and Szeptycki showed that $\mathcal{L}$ is not a Q-space [12].

In the following theorem, we prove a stronger condition than nonseparability for Lindelöf Q-spaces of certain cardinality.

Theorem 3.2. *Let X be a T_3, Lindelöf Q space, such that $w(X) \leq \mathfrak{c}$. Then $|X| < cf(\mathfrak{c})$.*

The previous theorem has several consequences regarding which Q-spaces cannot exist. Some of these are the following.

Corollary 3.3. *Let X be a T_3 topological space with $|X| \geq cf(\mathfrak{c})$. Then:*

(i) if X is Lindelöf and $\chi(X) \leq \mathfrak{c}$, then X is not a Q-space;

(ii) if X is Lindelöf and locally compact, then X is not a Q-space;

(iii) if X is compact, then X is not a Q-space;

(iv) if X is countably compact, then X is not a Q-space.

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