Directed studies - Eigenvalues of Cayley Graphs

Györgypál Tamás
Supervisor: Somlai Gábor

In this note, we present fundamental definitions and techniques concerning tiling sets and pairs of groups. Then we go through how the irreducible representations of $\mathrm{SL}(2,p)$ can be constructed, and finally, we discuss a recent question formulated by Sanming Zhou and Binzhou Xia.

Tiling sets of classical groups

First, let us see how one can define tiling sets and pairs in an arbitrary group.

Definition 1 (Tiling set). Let G be an arbitrary group. $A \subset G$ is a tiling set of G if there is a tiling pair $B \subset G$ such that every element $g \in G$ can be expressed uniquely as the product $g = ab$, where $a \in A$ and $b \in B$.

Fix a symmetric generating set S of a finite group G and consider the Cayley graph $Cay(G,S)$. It is easy to see that the eigenvalues of this graph are all real. If A is the adjacency matrix of $Cay(G,S)$, then

$$A = \sum_{s \in S} \rho(s),$$

where ρ is the regular representation of G. Furthermore, the irreducible components of ρ are every irreducible representation of G with the multiplicity of their dimensions, so

$$\rho = \bigoplus_{\varphi \in \widehat{G}} \underbrace{\varphi \oplus \cdots \oplus \varphi}_{d_\varphi},$$

where d_φ is the dimension of the irreducible representation φ.

If a nontrivial subset $M \subset G$ has no 0 eigenvalues in any of the irreducible representations (we are talking about the matrices $\sum_{m \in M} \varphi(m)$, where φ is an irreducible representation), then it cannot be a tiling set because the matrices of its tiling pair would have to be the zero matrices in every irreducible representation. This is because, in the language of group algebras,

$$\Big(\sum_{a \in A} a\Big)\Big(\sum_{b \in B} b\Big) = \sum_{g \in G} g$$

implies that

$$\Big(\sum_{a \in A} \varphi(a)\Big)\Big(\sum_{b \in B} \varphi(b)\Big) = \sum_{g \in G} \varphi(g),$$

but for any non-trivial irreducible representation, the sum of the representations of every group element is the zero matrix. We assumed that $\sum_{a\in A}\varphi(a)$ has no zero eigenvalues, so it is invertible, which means that $\sum_{b\in B}\varphi(b)$ has to be the zero matrix. By the Wedderburn-Artin theorem, $\sum_{b\in B} b$ has to be zero.

It is a direct consequence of Selberg's work [6] or Bourgain-Gamburd's [1] that the family of graphs $Cay\big(\mathrm{SL}(2,p),S\big)$ forms an expander where p goes through the prime numbers and

$$S=\left\{\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}1&-1\\0&1\end{pmatrix},\begin{pmatrix}1&0\\1&1\end{pmatrix},\begin{pmatrix}1&0\\-1&1\end{pmatrix}\right\}.$$

Let s_1, s_1^{-1}, s_2 and s_2^{-1} denote these four matrices in order. Recently, Sanming Zhou and Binzhou Xia asked whether this symmetric set S in $\mathrm{SL}(2,p)$ is a tiling set or not. Obviously, we can get different answers for different primes.

First, we have to understand the irreducible representations of $\mathrm{SL}(2,p)$; these are usually divided into two classes: the principal series and the discrete series representations. Before we get into the construction of these representations, we recall how an induced representation can be obtained. If $H\le G$ is a subgroup and η is a representation of H in a vector space V_η, then the induced representation of G with respect to η may be obtained as follows: Let $\mathrm{Ind}(V_\eta)$ denote the vector space of functions $f: G\to V_\eta$ such that

$$f(hx)=\eta(h)f(x)$$

for all $h\in H$ and $x\in G$. There is a representation of G on $\mathrm{Ind}(V_\eta)$ by right translation,

$$\big(\varphi(g)f\big)(x)=f(xg).$$

This is called the representation of G induced from H by η. The dimension of the induced representation is $[G:H]d_\eta$, where d_η is the dimension of V_η.

The Principal Series

This class of representations of $\mathrm{SL}(2,p)$ is constructed by inducing some one dimensional representations of the subgroup of upper triangular matrices

$$B=\left\{\begin{pmatrix}k&u\\0&k^{-1}\end{pmatrix}\;\middle|\; u\in\mathbb{F}_p, k\in\mathbb{F}_p^\times\right\}.$$

Namely, consider the representations ψ_j $(0\le j\le p-2)$ of B, defined by

$$\psi_j\left(\begin{pmatrix}k&u\\0&k^{-1}\end{pmatrix}\right)=\omega^{jk},$$

where $\omega=e^{\frac{2\pi i}{p-1}}$ is a $(p-1)$-th primitive root of unity. It is easy to check that this is indeed a representation. Basically, one can simply say that $\{\psi_j \mid 0\le j\le p-2\}$ are the irreducible characters of the group $\mathbb{F}_p^\times$. Let φ_{ψ_j} denote the induced representation of ψ_j to the whole group $\mathrm{SL}(2,p)$.

Theorem 1. (See [4, Theorem 2.1.])

(1) Let ψ_j and ψ_l be irreducible representations of $\mathbb{F}_p^\times$. Suppose that $\psi_j^2, \psi_l^2 \neq 1$, then φ_{ψ_j} and φ_{ψ_l} are irreducible representations (of dimension $p+1$). Furthermore, φ_{ψ_j} and φ_{ψ_l} are equivalent if and only if $\psi_j = \psi_l$ or $\psi_j^{-1} = \psi_l$.

(2) It is known that $\psi_{\frac{p-1}{2}}$ is the only non-trivial quadratic representation of $\mathbb{F}_p^\times$, meaning $\psi_{\frac{p-1}{2}}^2 = 1$. Its induced representation to $\mathrm{SL}(2,p)$ splits in the direct sum of two inequivalent irreducible representations, each of degree $\frac{p+1}{2}$.

(3) φ_1 is equivalent to the direct sum of the trivial representation of $\mathrm{SL}(2,p)$ and an irreducible representation of degree p, which is called the Steinberg representation.

The good news is that by choosing the coset representatives thoughtfully, these representations are easier to understand, and one can think of them as Möbius transformations on the projective line $\mathbb{P}^1(\mathbb{F}_p)$. One possible choice of the coset representatives is

$$\ldots, x_u = \begin{pmatrix} 0 & 1 \\ -1 & -u \end{pmatrix}, \ldots, x_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let's fix a character ψ of $\mathbb{F}_p^\times$. In this way, the representations of our four generating elements can be considered as the actions $\varphi_\psi(s_1), \varphi_\psi(s_1^{-1}), \varphi_\psi(s_2), \varphi_\psi(s_2^{-1}) : \mathbb{P}^1(\mathbb{F}_p) \to \mathbb{P}^1(\mathbb{F}_p)$,

$$x \overset{\varphi_\psi(s_1)}{\mapsto} x+1, \; x \overset{\varphi_\psi(s_1^{-1})}{\mapsto} x-1,$$

$$x \overset{\varphi_\psi(s_2)}{\mapsto} \begin{cases} \psi(1+x)\cdot \frac{x}{1+x} & \text{if } x \in \mathbb{P}^1(\mathbb{F}_p) \setminus \{-1,\infty\} \\ \psi(-1)\cdot\infty & \text{if } x = -1 \\ 1 & \text{if } x = \infty \end{cases},$$

$$x \overset{\varphi_\psi(s_2^{-1})}{\mapsto} \begin{cases} \psi(1-x)\cdot \frac{x}{1-x} & \text{if } x \in \mathbb{P}^1(\mathbb{F}_p) \setminus \{1,\infty\} \\ \infty & \text{if } x = 1 \\ \psi(-1) & \text{if } x = \infty \end{cases}.$$

For example, if $p=3$, then the matrix representatives of our generating elements will be

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \psi(2) & 0 & 0 \\ 0 & 0 & \psi(2) & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \psi(2) & 0 \\ 0 & 0 & 0 & \psi(2) \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

and their sum

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 0 & 1+\psi(2) & 1 \\ 1 & 1+\psi(2) & 0 & \psi(2) \\ 0 & 1 & \psi(2) & 2 \end{pmatrix}.$$

The Discrete Series

The construction of the representations in the discrete series is more complicated. Usually, it is done by defining the representation on the Weyl element w, which is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and on the elements of the form

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix}.$$

For details, see [5]. We consider the field $\mathbb{F}_{p^2} = \{a + b\sqrt{\alpha} \mid a, b \in \mathbb{F}_p\}$, where $\alpha \in \mathbb{F}_p$ and $\nexists x \in \mathbb{F}_p : x^2 = \alpha$. The norm function is:

$$\mathcal{N} : \mathbb{F}_{p^2} \to \mathbb{F}_p$$

$$a + b\sqrt{\alpha} \mapsto a^2 - b^2\alpha$$

It's the same norm function that is usually defined as $\mathcal{N}(x) = x^{p+1}$. It is known that $T^1 = \{x \in \mathbb{F}_{p^2} \mid \mathcal{N}(x) = 1\} \cong Z_{p+1}$. T^1 has $p+1$ irreducible characters, each implying a representation of $\mathrm{SL}(2,p)$ from the discrete series representations. There is only one non-trivial quadratic character (just as we have seen in the case of the principal series), apart from that, every character will imply an irreducible representation. The one induced by the non-trivial quadratic character will split into two inequivalent irreducible representations of degree $\frac{p-1}{2}$ (see [4, Theorem 2.3.]). Let's fix one character θ. Let $\varepsilon = e^{\frac{2\pi i}{p}}$ and for each $t \in \mathbb{F}_p^\times$, we fix an element $v_t \in \mathbb{F}_{p^2}$ such that $\mathcal{N}(v_t) = t$ (basically, these will be the basis of our vector space). In this way, the matrices of s_1, s_1^{-1}, s_2 and s_2^{-1} will be

$$\begin{pmatrix} \varepsilon & 0 & \dots & 0 \\ 0 & \varepsilon^2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \varepsilon^{p-1} \end{pmatrix}, \begin{pmatrix} \varepsilon^{-1} & 0 & \dots & 0 \\ 0 & \varepsilon^{-2} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \varepsilon^{-(p-1)} \end{pmatrix},$$

$$\Big(-\theta(-1) \cdot \varepsilon^{x+y} \cdot \frac{1}{p} \sum_{u \in T_1} \theta(u) \cdot \varepsilon^{\mathrm{Tr}(v_y \cdot \overline{v_x} \cdot u)} \Big)_{x,y} \text{ and } \Big(-\varepsilon^{-(x+y)} \cdot \frac{1}{p} \sum_{u \in T_1} \theta(u) \cdot \varepsilon^{\mathrm{Tr}(v_y \cdot \overline{v_x} \cdot u)} \Big)_{x,y},$$

where $\mathrm{Tr}(a + b\sqrt{\alpha}) = 2a$. So, the sum of these is

$$\Big((\varepsilon^x + \varepsilon^{-x}) \cdot \delta_{x,y} - \big(\theta(-1)\varepsilon^{x+y} + \varepsilon^{-(x+y)}\big) \cdot \frac{1}{p} \sum_{u \in T_1} \theta(u) \cdot \varepsilon^{\mathrm{Tr}(v_y \cdot \overline{v_x} \cdot u)} \Big)_{x,y}.$$

Conjectures

I wrote a program to calculate the determinant of the sum matrix symbolically for every possible irreducible representation to see whether 0 appears as an eigenvalue or not.

It is worth mentioning that the trivial and Steinberg representations were treated together (since the representation derived from the trivial character splits into these two) and also the quadratic ones in both cases (the principal and the discrete series).

The representations in the principal series are much easier to realize, as we have seen before, and I could calculate the determinants up to $p = 107$, but those in the discrete series proved to be more difficult. There, my computational limit was the $p = 37$ case. Up to 37, the only primes for which every matrix has a non-zero determinant are 7, 19 and 29. For example, for $p = 7$, the determinants are:

Principal Series		Discrete Series	
Character	Determinant	Character	Determinant
Trivial ($k = 0$)	-64	Order 8 ($k = 1, 7$)	4
Order 6 ($k = 1, 5$)	64	Order 4 ($k = 2, 6$)	4
Order 3 ($k = 2, 4$)	-64	Order 8 ($k = 3, 5$)	4
Quadratic ($k = 3$)	4	Quadratic ($k = 4$)	36

One can find fascinating that in the above example every determinant is an integer. Obviously, in general the values are polynomials in ω, where $\omega = e^{\frac{2\pi i}{p-1}}$ for the principal series and $\omega = e^{\frac{2\pi i}{p+1}}$ for the discrete series. However, there are some interesting possible patterns in the results I could not prove yet.

Conjecture 1. Let $p \geq 3$ an arbitrary prime. In a representation of the principal series induced from a maximal order character of Z_{p-1}, the determinant of the sum matrix will be $2^{2\lfloor \frac{p}{6} \rfloor + 4}$.

Conjecture 2. Let $p \geq 3$ an arbitrary prime. In a representation of the discrete series induced from a maximal order character of Z_{p+1}, the determinant of the sum matrix will be $2^{2\lfloor \frac{p}{6} \rfloor}$.

It seems that the number of representations for which the determinant of the sum matrix is 0 is small enough, but further work is needed to be done to precisely characterize the representations for which this occurs.

References

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