

Sphere Eversion

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1. Introduction

The topic of my research was sphere eversion. By this, we mean embedding a spheroid of a given dimension into a Euclidean space of the same dimension, and then, through immersions, making its "exterior" become its "interior." So let $k < n$ and $f : S^k \rightarrow \mathbb{R}^n$ be a natural embedding of S^k into \mathbb{R}^n . We invert this embedded sphere by a regular homotopy, that is, we cause the orientation of our sphere to be reversed. Here are the definitions to make this understandable.

Definition: Let M and N be two differentiable manifolds, and let $f : M \rightarrow N$ be a differentiable map. We say that the differentiable map f is an immersion if the following map

$$D_m f : T_m \in M \rightarrow T_{f(m)} \in N$$

is injective for every $m \in M$. In other words, we say that f is an immersion if the rank of the Jacobi matrix associated with f is equal to the dimension of the manifold M at every point.

Definition: Let M and N continue to be two differentiable sets, and let f and g be two immersions from M to N . The immersions f and g are regularly homotopic if there exists a homotopy

$$H : M \times [0, 1] \rightarrow N$$

that is an immersion at every coordinate $t \in [0, 1]$, and it holds that $H(m, 0) = f(m)$, for all $m \in M$ and $H(m, 1) = g(m)$ for all $m \in M$. We call this H a regular homotopy.

Last semester, I presented an eversion of S^2 in \mathbb{R}^3 . This time, I will present an eversion of S^2 in $\mathbb{R}P^2$ that can also be realized in Euclidean space.

If our goal is to everse S^2 in $\mathbb{R}P^2$, then the task is straightforward. This is achieved by identifying $\mathbb{R}P^3$ with $B^3 \cup N(\mathbb{R}P^2)$. Here, $N(\mathbb{R}P^2)$ denotes the twisted line bundle

over $\mathbb{R}P^2$, whose boundary we attach to the three-dimensional solid sphere, B^3 . Without going into detail, the procedure involves embedding an S^2 into B^3 in the standard way so that its center coincides with the center of B^3 , then we begin to “push it outward” all the way to the boundary of B^3 and then “drag it along” $N(\mathbb{R}P^2)$.

In order to illustrate the inversion that can be realized in \mathbb{R}^3 , I need to explain a few concepts.

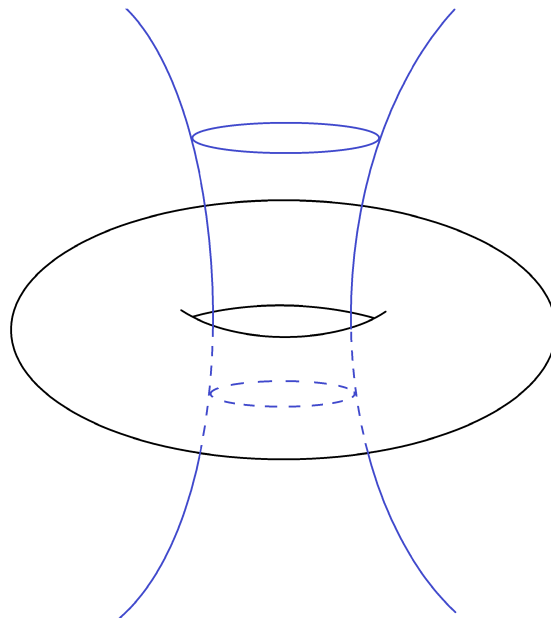
1.1 Heegaard splitting

Let M and N be two handlebodies, and let M and N have the same genus. Let $\alpha : \partial M \rightarrow \partial N$ be a homeomorphism. It is also true that α is a direction-changing map. Here, ∂ denotes the topological boundary of M and N . After this, glue together M and N along α .

$$P := M \cup_{\alpha} N$$

With this we get an orientable 3-manifold.

What is important from my perspective is the so-called standard genus one splitting of S^3 . We can imagine S^3 as the unit sphere in \mathbb{C}^2 . The points where each coordinate has norm $\frac{1}{\sqrt{2}}$ form the so-called Clifford torus. We can split S^3 with two solid torus glued along the Clifford torus.



1.2 Hopf flow

The Hopf fibration is perhaps one of the best-known fibrations. Its total space is S^3 , its base space is $\mathbb{C}P^1 \cong S^2$, and its fiber is S^1 .

$$\begin{array}{c}
S^3 \leftarrow S^1 \\
\downarrow \\
S^2
\end{array}$$

We can identify $S^3 \subset \mathbb{C}^2$ with the following set:

$$S^3 = \{(\alpha, \beta) \in \mathbb{C}^2 \mid \alpha\bar{\alpha} + \beta\bar{\beta} = 1\}$$

We define the Hopf flow as the following action:

$$e_{\text{Hopf}} : S^3 \rightarrow S^3$$

$$e_{\text{Hopf}}^{i\theta}(\alpha, \beta) = (e^{i\theta}\alpha, e^{i\theta}\beta)$$

This action creates orbits which are the fibers of the Hopf fibration. As we saw before, we have the standard genus one splitting of S^3 . In this splitting the Clifford torus can be described by the following set:

$$CT = \{(\alpha, \beta) \in \mathbb{C}^2 \mid \alpha\bar{\alpha} = \beta\bar{\beta}\}$$

where CT denotes the Clifford torus. Another important property of the Hopf flow is that it is invariant under the two handlebodies. We will see this with parametrized handlebodies (solid torus) in the eversion.

2. The construction of the eversion

First, I will present a few notations. As we saw earlier in the genus one splitting of S^3 there is the Clifford torus which determines two complementary solid torus. Lets call the first one SD and the second one SD^* . These solid toruses define two also complementary core circles which can be parametrized with the following sets: $\{(e^{i\phi}, 0)\}$ and $\{(0, e^{i\psi})\}$. These two solid toruses give the genus one Heegaard decomposition. In CT we define two oriented circles: $\lambda := \left(\frac{e^{i\phi}}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\mu := \left(\frac{1}{\sqrt{2}}, \frac{e^{i\phi}}{\sqrt{2}}\right)$. These two circles form a homology basis of $H_1(CT, \mathbb{Z})$. This means that one of them is homotopic to the meridian of CT and the other is homotopic to the longitude of CT . So with these notations we can describe SD and SD^* in the following way: $SD \cong \lambda \times D$, $SD^* \cong D^* \times \mu$ where D and D^* are 2-disks and $\partial D = \mu$, $\partial D^* = \lambda$.

We define the $e^{i\psi}$ action: $e^{i\psi}(\alpha, \beta) := (\alpha, e^{i\psi}\beta)$. By this, we define $D_\psi^* := (D^*, e^{i\psi})$. Clearly $D_\psi^* \subset SD^*$ and $\partial D_\psi^* := K_\psi^* \subset CT$ ($K_0^* = \lambda$).

I mentioned earlier that this eversion will be realized in \mathbb{RP}^3 . The easiest way to think about \mathbb{RP}^3 is by taking the factor of S^3 by its antipodal points.

We will lift our regular homotopy in a way that it will remain regular. This homotopy starts with a small sphere \bar{S}_* which we put to the center of D_0^* . Then we radially expand it until it becomes tangent to CT (\bar{S}_0^2). With \bar{S}_ψ^* we denote a further expanded sphere for which is true that it intersects CT in K_+^* and K_-^* . K_+^* and K_-^* bound an annulus (\bar{A}_ψ^*) in the other solid torus (SD). With this annulus we can decompose our \bar{S}_ψ^* into three parts:

$$\bar{S}_\psi^* = \bar{D}_\psi^2 \cup \bar{A}_\psi^* \cup \bar{D}_{-\psi}^2$$

$K_{+\psi}^*$ and $K_{-\psi}^*$ also bound the disks $D_{-\psi}^*$ and $D_{+\psi}^*$. These disks intersect CT orthogonally. Furthermore we can choose an annulus A_ψ^* , meeting CT orthogonally along K_\pm^* . So we can decompose $S_\psi^* := D_\psi^* \cup A_\psi^* \cup D_{-\psi}^*$. This method can be familiar in this topic, last semester I presented a construction where was also a step where we modified our annulus such that it became orthogonal. It is clear that each S_ψ^* is the boundary of a 3-ball neighbourhood of D_0^* and there is a natural collapse of these spheres into D_0^* . These spheres creates an eversion in \mathbb{RP}^3 . Our goal is to modify this eversion such that it will remain an eversion after we lift it up to S^3 .

Remark: We want our eversion to be realized in \mathbb{R}^3 . In our construction we worked with S^3 instead of \mathbb{R}^3 but we know that $S^3 \cong \mathbb{R}^3 \cup \{\infty\}$, so S^3 is the compactification of \mathbb{R}^3 .

If we look at the definition of the Hopf flow it is clear that SD , SD^* and CT are invariant under the this action. They are also invariant under the antipodal map $a : S^3 \rightarrow S^3$. This action defines circles in S^3 (the orbits of the actions are circles). Let $P := \lambda \cap \mu$. If we take the orbit of P , it intersects K_ψ^* at $P_\psi^* := \left(\frac{e^{i\psi}}{\sqrt{2}}, \frac{e^{i\psi}}{\sqrt{2}}\right)$. D_θ^h denotes the rotated disk $e_{\text{Hopf}}^{i\theta}(D_\theta^*)$. As the point P follows P_θ^* , it induces a reparametrization of A_θ^* , lets call it A_θ^h .

Without going into details, there is a reparametrization of SD such that the Hopf flow will look like the following: $e_{\text{Hopf}}^{i\theta}(\alpha, \beta) = (e^{i\theta}\alpha, \beta)$. From now we will use the $e_{\text{rot}}^{i\theta}$ notation because with this reparametrization it rotates in the first coordinate. So let $D_\theta^h := e_{\text{rot}}^{i\theta}(D_0^h)$ and $S_\theta^h = D_\theta^h \cup A_\theta^h \cup D_{-\theta}^h$. Furthermore, let $S_\theta^2 = D_{-\theta}^2 \cup A_{\theta/2}^2 \cup D_\theta^2$ be their projection to \mathbb{RP}^2 . The SD/a embeds in \mathbb{R}^3 realizing the Hopf action as rotation around the Z -axis and the attaching circle arising from SD^*/a will become a $(2, -1)$ torus which bounds a disk D_0 immersed in \mathbb{R}^3 . So the other disks D_θ will be embedded by rotating D_0 .

References

- [1] Aitchison, Iain R. "TheHoliverse': holistic eversion of the 2-sphere in R^3 ." arXiv preprint arXiv:1008.0916 (2010).
- [2] Smale, Stephen. "The classification of immersions of spheres in Euclidean spaces." *Annals of mathematics* 69.2 (1959): 327-344.

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Task	Used tool	Location of use	Remarks
Grammar check	Writefull	Entire note	—
Formatting the note	Gemini 3.1 Pro	Entire note	—

Other than the ones listed above, I did not use any other artificial intelligence-based tools.