Dwork's theorem on the zeta-functions of affine hypersurfaces

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Introduction

Definition

Let X be an vaffine hypersurface defined by the polynomial f in n variables with coefficients in $\mathbb{F}_q$. Define N_s as the number of points in X over $\mathbb{F}_{q^s}$. With these notations the zeta-function of X is:

$$\zeta(X/F_q, t) = exp\left(\sum_{s=1}^{\infty} N_s \frac{t^s}{s}\right)$$

Theorem 1.0.2

Theorem 1.0.2 (Dwork). The zeta-function of any affine (or projective) hypersurface is a rational function.

Characters as a power series

We use that for $a \in \mathbb{F}_q$ and a pth unit root $\varepsilon \in \mathbb{C}_p$, if t is the Teichmüller lift of a then:

$$a \mapsto \varepsilon^{\operatorname{Tr} a} = \varepsilon^{\operatorname{Tr}_{\mathbb{Q}_q} t}$$

is a character.
Also $\operatorname{Tr}_{\mathbb{Q}_q} t = t + t^p + \ldots + t^{p^{s-1}}$.
To express this character as a power series in t the binomial expansion would be naive approach, but in $\mathbb{C}_p$ this does not converge on the closed unit disk, so neither does in t.

Characters as a power series

The solution is to use instead of the binomial expansion the following power series:

$$F(x,y) = (1+y)^x (1+y^p)^{(x^p - x)/p} (1+y^{p^2})^{(x^{p^2} - x^p)/p^2} \ldots (1+y^{p^n})^{(x^{p^n} - x^{p^{n-1}})/p^n} \ldots .$$

And for

$$\theta(x) = F(x, \varepsilon - 1)$$

it is easy to see that

$$\varepsilon^{\mathrm{Tr}\, a} = \varepsilon^{\mathrm{Tr}_{\mathbb{Q}_q}\, t} = \theta(t)\theta(t^p)\ldots\theta(t^{p^s}).$$

The trace formula

Definition

Let R denote the set of formal power series over $\mathbb{C}_p$ in n variables, and U the set of ordered n-tuples of nonnegative integers. Define the following two $\mathbb{C}_p-$linear transformations on R:

$$L_G(r) = Gr, \text{ where } G \in R,$$

$$r = \sum_{u \in U} a_u x^u \mapsto T_q(r) = \sum_{u \in U} a_{qu} x^u, \text{ where } q \in \mathbb{Z}^+.$$

Then let $\Psi_{q,G} = T_q \circ L_G$.

The trace formula

Definition

$$R_0 = \left\{ G = \sum_{w \in U} g_w x^w \in R : \exists M > 0, \quad \mathrm{ord}_p g_w \geq M|w| \quad \forall w \in U \right\}.$$

Lemma

Let $G \in R_0$. Then $\mathrm{Tr}\left(\Psi_{q,G}^s\right)$ converges for $s \in \mathbb{Z}^+$, and

$$(q^s - 1)^n \mathrm{Tr}\left(\Psi_{q,G}^s\right) = \sum_{\substack{x \in \mathbb{C}_p^n \\ x^{q^s - 1} = 1}} G(x) G(x^q) \ldots G(x^{q^s - 1}).$$

The determinant

We can extend the notion of $det(1 - At)$, where t is an indeterminate, to infinite A matrices as a formal power series $\det(1 - At) = \sum_{m=0}^{\infty} b_m t^m$.

Lemma

If $G(x) = \sum_{w \in U} g_w x^w \in R_0$ and $\Psi = T_q \circ L_G$, so that Ψ has a matrix $A = \{g_{qv-u}\}_{u,v \in U}$, then the series $\det(1 - At)$ is a well-defined power series over $\mathbb{C}_p$ with infinite radius of convergence, and

$$\det(1 - At) = \exp_p \left(- \sum_{s=1}^{\infty} \mathrm{Tr}\,(A^s) t^s / s \right).$$

Analytic expression of the zeta function

It is enough to see for N_s': the number of zeros of f with nonzero coordinates.Here we use that the character derived from the trace of elements in $\mathbb{F}_{q^s}$ is non-trivial, so it is an indicator of whether an element $u \in \mathbb{F}_{q^s}$ is 0:

$$\sum_{x_0 \in \mathbb{F}_{q^s}^\times} \varepsilon^{\mathrm{Tr}\,(x_0 u)} = \begin{cases} -1, \text{ if } u \in \mathbb{F}_{q^s}^\times \\ q^s - 1, \text{ if } u = 0. \end{cases}$$

Substituting $u = f(x_1, ..., x_n)$ this allow us to calculate:

$$q^s N_s' = (q^s - 1)^n + \sum_{\substack{x_0, \dots, x_n \in \mathbb{C}_p \\ x_0^{q^s-1} = \dots = x_n^{q^s-1} = 1}} \prod_{i=1}^{N} \Theta([a_i]x^{w_i}) \dots \Theta([a_i]^{p^{rs-1}} x^{p^{rs-1} w_i}).$$

But for a proper G product of some of the Θ terms this equals to:

$$(q^s - 1)^n + \sum_{\substack{x_0, \dots, x_n \in \mathbb{C}_p \\ x_0^{q^s-1} = \dots = x_n^{q^s-1} = 1}} G(x) \dots G(x^{q^{s-1}}) = (q^s - 1)^n + (q^s - 1)^{n+1} \mathrm{Tr}\,(\Psi_{q,G}^s).$$

Analytic expression of the zeta function

This gives:

$$\zeta'(X/\mathbb{F}_q, t) = \prod_{i=1}^{n} (1 - q^{n-i-1} t)^{-1^{i+1} \binom{n}{i}} \prod_{i=0}^{n+1} \det(1 - A q^{n-i} t)^{-1^{i+1} \binom{n+1}{i}}$$

where A is the matrix of $\Psi_{q,G}$.

Final steps

The end of the proof is rather technical, mostly linear algebra. It relies on the Weierstrass preparation theorem.