Directed studies

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1 Introduction

We continue our analysis of the Milnor fibration, following [2]. Specifically we inspect the Milnor fiber from a homology theory perspective. To this end first, we summarize the prerequisites and introduce basic concepts in homology.

2 The Milnor fibration

Generally $\mathbb{K}^n$ is used, where either $\mathbb{C}^n$ or $\mathbb{R}^n$ would be meaningful. The objects of interest are algebraic sets, hypersurfaces and their singularities to be more exact.

Definition 2.1. We call $V \subset \mathbb{K}^n$ an algebraic set, if it is the common zero set of a collection of $f_k \in \mathbb{K}[x_1, \ldots, x_n]$ polynomials.

A algebraic set $V \subset \mathbb{K}^n$ can be identified bijectively with the radical ideal in $\mathbb{K}[x_1, \ldots, x_n]$ which it is the zero set of. From this arises many characterization of algebraic sets using ring theoretic tools in $\mathbb{K}[x_1, \ldots, x_n]$. Example being that the Krull dimension of $\mathbb{K}[x_1, \ldots, x_n]/I$ (the so called coordinate ring of $V(I)$) for a radical ideal $I \triangleleft \mathbb{K}[x_1, \ldots, x_n]$ is the "intuitive" dimension of the associated algebraic set denoted $V(I)$.
The case is similar in a local setting, where we deal with algebraic set germs around a given point. This point of interest is usually chosen to be an isolated point of singularity.

Definition 2.2. A point $x \in V$ is called non-singular if the matrix $(\partial_j f_i(x))_{ij}$ attains full rank (as in maximal for x), and singular if not.

Our way of analysis is to take the intersection of an algebraic set, with a small enough sphere around such an isolated singularity. This kind of examination is supported by the following theorem.

Theorem 2.3. *Given a $V \subset \mathbb{K}^n$ algebraic set, and $x \in V$ isolated singularity or a regular point. Then for a small enough ε, the cone over $S_\varepsilon \cap V$ with base point x is equivalent in the following sense to $D_\varepsilon \cap V$:*

$$(D_\varepsilon, \{x + t(y - x) \in \mathbb{K}^n : \forall t \in [0,1], \forall y \in S_\varepsilon \cap V\}) = (D_\varepsilon, Cone(S_\varepsilon \cap V)) \simeq (D_\varepsilon, D_\varepsilon \cap V)$$

Here the equivalence is meant to be understood as homeomorphism between embedded topological pairs. We call the set $K = S_\varepsilon \cap V \subset \mathbb{K}^n$ (which is unique for small ε) the link of the singularity x. In the trivial example, when x in a non-singular point, K is simply an unknotted sphere.

In the case of surfaces in $\mathbb{C}^2$ (which are also curves) these links can be completely classified as interlinked iterated toric knots. (Hyper-) surface meaning 1 codimensional, curve meaning 1 dimensional algebraic set.
Using the link of a singularity of a hypersurface in $\mathbb{C}^n$ one can decompose the small $S^{2n-1} \cong S_\varepsilon$ into $S_\varepsilon \setminus K$ and K obviously. However the following theorem of Milnor allows us to further decompose and analyze $S_\varepsilon \setminus K$.

Theorem 2.4. *Let $V(f) \in \mathbb{C}^n$ be a hypersurface, with an isolated singularity z. Then for a small enough ε, with $K = S_\varepsilon \cap V(f)$ (the link of the isolated singular point z) $S_\epsilon \setminus K$ has a locally trivial fibration over S^1 with*

$$\Phi : S_\varepsilon \setminus K \to S^1, \quad \Phi(z) = \frac{f(z)}{|f(z)|}$$

as projection map.

We denote a fiber $F_\theta := \Phi^{-1}(\theta)$, where S^1 is parametrized as the complex unit circle $\{e^{i\theta} : \theta \in [0, 2\pi)\} \subset \mathbb{C}$. Such a fiber is a smooth manifold of $2n-2$ real dimension, that has the link as its boundary, $\partial F_\theta = K$. This in itself is an interesting result, nevertheless we would like to continue further towards some characterization using these fibers. As it happens, continuation requires some homology theory.

Basic homology

Homology theory allows us to turn questions of topology into questions of algebra. Similar to homotopy groups, one can construct (co-)homology groups. However homology groups are much easier to compute in general. Let us begin with a short description of singular homology.

Definition 2.5. Let us use $\triangle^n$ instead of the n-dimensional simplex (the triangle for $n = 2$, the tetrahedron for $n = 3, \ldots$). For a given manifold M we call the continuous maps $f : \triangle^n \to M$ singular n simplexes. $C_n(M; \mathbb{Z})$ denotes the free abelian group generated by the singular n simplexes of M. That is:

$$C_n(M) := \left\{ \sum_{f \text{ a singular } n \text{ simplex}} k_f f : \text{only finitely many } k_f \in \mathbb{Z} \text{ are nonzero} \right\}.$$

The next step is to construct so called boundary maps between such groups of neighboring indices. $\partial_n : C_n(M; \mathbb{Z}) \to C_{n-1}(M; \mathbb{Z})$ homomorphism is defined as follows.

$$\partial_n(f) := \sum_{i=1}^{n} (-1)^i f|[v_1, \ldots, \hat{v_i}, \ldots, v_n]$$

Where $f|[v_1, \ldots, \hat{v_i}, \ldots, v_n]$ is f restricted to the side of the n simplex that does not contain the i^{th} vertex v_i, which is an $n-1$ simplex, and hence the definition makes sense. This can be linearly extended to the entirety of $C_n(M; \mathbb{Z})$.

A crucial property of the boundary maps is that the composition of two appropriate ones is the zero map.

$$\partial_{n-1} \circ \partial_n \equiv 0$$

Meaning, the $C_n(X;\mathbb{Z})$-s form a half-exact sequence:

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1}(X;\mathbb{Z}) \xrightarrow{\partial_{n+1}} C_n(X;\mathbb{Z}) \xrightarrow{\partial_n} C_{n-1}(X;\mathbb{Z}) \xrightarrow{\partial_{n-1}} \cdots,$$

a chain-complex. Hence the elements of $C_n(X;\mathbb{Z})$ are referred to as n-chains. It is because of this half exactness, that we can define the n^{th} homology group:

$$H_n(X;\mathbb{Z}) \coloneqq \ker \partial_n / \operatorname{im} \partial_{n+1}$$

We call the elements of $\ker \partial_{n-1}$ cycles, the elements of $\operatorname{im} \partial_n$ boundaries. Note that $\mathbb{Z}$ could be replaced with an arbitrary abelian group. A fundamental result connecting the homotopy groups and homology groups of topological spaces is the Hurewicz theorem.

Theorem 2.6. *Let X be $n-1$ connected for $n > 1$, meaning $\pi_j(X)$ are trivial for $j < n$. Then $H_j(X;\mathbb{Z})$ is trivial for $j < n$, and the first not necessarily trivial homolgy group is isomorphic to the corresponding homotpy group. That is:*

$$H_n(X;\mathbb{Z}) \cong \pi_n(X)$$

In the case of $n = 1$, the first homology group is isomorphic to the abelianization of the fundamental group:

$$H_1(X;\mathbb{Z}) \cong \pi_1(x)/[\pi_1(X), \pi_1(X)]$$

There exists a dual notion to homology, cohomology namely. In this case the chain complex is as follows.

$$\cdots \xleftarrow{\partial^{n+2}} Hom(C_{n+1}(X;\mathbb{Z}),\mathbb{Z}) \xleftarrow{\partial^{n+1}} Hom(C_n(X;\mathbb{Z}),\mathbb{Z}) \xleftarrow{\partial^n} \cdots,$$

Here the boundary maps are induced by ∂_i-s, and the cohomology groups are defined similarly, and denoted $H^n(M;\mathbb{Z})$. Put concisely, we apply the contravariant functor $Hom(.,\mathbb{Z})$ to the homology chain complex. A famous result connecting homology groups and cohomolgy groups is Poincaré's duality theorem.

Theorem 2.7. *If M is an n-dimensional orientable manifold, then*

$$H_{n-j}(M;\mathbb{Z}) \cong H^j(M;\mathbb{Z})$$

As an example, which we will use later, homology groups allow us to formalize the notion of the number of "n-dimensional holes". A hole can be thought of as a sphere or hollow simplex, that cannot be "filled in". Or, somewhat more formally, a sphere or hollow simplex for which there is some obstruction to attaching the interior. Such object directly correspond to cycles, of which no multiple is a boundary, therefore these create non-trivialities (generate a $\mathbb{Z}$) in homology groups.

Example 2.8. Let us look at S^1. Intuitively one would say, a circle has one hole. This is reflected in the homology groups, which are:

$$H_n\left(S^1;\mathbb{Z}\right) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

$H_1(S^1;\mathbb{Z})$ has a free component of rank 1, meaning it has 1 "2-dimensional hole".

The number rank H_0 correspond to the connected components of a manifold. Since a 1-simplex is a line segment, a 0-simplex is a point, two points that cannot be connected with a line segment generate separate (free) components in H_0. With this we are ready to define Betti numbers.

Definition 2.9. For a given manifold M, let the n^{th} Betti number, b_n denote the rank of free component in the abelian group $H_n(M;\mathbb{Z})$.

Remark. A homology group is not necessarily torsion free. $H_1(\mathbb{R}P^2;\mathbb{Z}) = \mathbb{Z}_2$, and so the first Betti number of $\mathbb{R}P^2$ is 0

Analysis of the Milnor fiber

Now let us return to the Milnor fibration, more specifically the Milnor fiber, F_θ. In the following section we assume that some deleted neighborhood of the singular point has no critical point of f, though the singularity might be a critical point.
As it was mentioned earlier at the end of section 2, F_θ is a smooth manifold of $2n-2$ real dimension, and has K as boundary. Our first statement is as follows.

Proposition 2.10. *The closure of F_θ, $\overline{F_\theta}$ as embedded in S_ε has the same homotopy type (has the same homotopy groups) as its complement, $S_\varepsilon \setminus \overline{F_\theta}$.*

This is not too difficult to see. The complement $S_\varepsilon \setminus \overline{F_\theta}$ is fibration over $S^1 \setminus \left\{e^{i\theta}\right\}$ and thus retracts onto a single fiber (preserving the homotopy groups). As these fibers are diffeomorphic, they have the same homotopy type.
Using this fact together with some Morse theory, one can prove a much more concrete statement.

Proposition 2.11. *Each fiber, F_θ is $n-2$-connected.*

The above proposition comes from the fact, that there exists a Morse function on F_θ, such that its critical points have indices $\leq n-1$. Therefore one can construct F_θ starting from a disk D^{2n-2} and attaching k-handles for $k \leq n-1$ inside S_ε. The complement, $S_\varepsilon \setminus D^{2n-2}$ is n connected, and attaching handles of index $\leq n-1$ cannot change the j^{th} homotopy group for $j \leq n-2$ in the complement. Thus, using proposition 2.10, we have the statement.
The handle decomposition also implies that F_θ has the homotopy type of an $n-1$-dimensional CW complex, and therefore $\pi_j(F_\theta) = 0$ for $j > n-1$. Our final description of the homotopy groups of the Milnor fiber is as below.

Theorem 2.12. *F_θ has same homotopy type as $S^{n-1} \vee \cdots \vee S^{n-1}$. In other words:*

$$\pi_j(F_\theta) = \begin{cases} \mathbb{Z}^k & \text{for } j = n-1 \text{ and } n \neq 2 \\ F_k & \text{for } j = 1 \text{ and } n = 2 \\ \text{trivial} & \text{otherwise} \end{cases},$$

for some $k \geq 0$ integer.

Remark. The separation of the first two cases is the result of the fact that π_1 is generally non-commutative, hence the free group of k generators is F_k, rather than $\mathbb{Z}^k$ as in the abelian case.

The question remains, what could k be, and how to characterize it. To this end, we introduce the multiplicity of a degenerate critical point.
First of all, $x^* \in \mathbb{C}^n$ is a degenerate critical point of f if the Hessian matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{ij}$ is singular at x^*. In order to measure the amount of degeneracy at a given point, we use the multiplicity of a solution to the set of polynomial equations:

$$\frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$$

Definition 2.13. In general, for a system of polynomial equations $g_1(x) = \cdots = g_n(x) = 0$, the multiplicity of a solution x^* is the degree of

$$x \mapsto \frac{g(x)}{\|g(x)\|}$$

from a small sphere around x^* to $S^{2n-1} \subset \mathbb{C}^n$. $g = (g_1, \ldots, g_n)$

In our case, given x^*, an isolated critical point of f and a solution to $f(x) = 0$, $x*$ will also be a solution to

$$\frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0.$$

Denote by μ the multiplicity of x^* as a solution to this.

Theorem 2.14. *The middle Betti number b_n of a fiber F_θ is equal to μ.*

Remark. This, in turn, answers our previous questions about k (the rank of the middle homotopy group of a fiber). By the Hurewicz theorem (2.6) the middle homotpy group is isomorphic to the middle homology group, $\pi_{n-1}(F_\theta) \cong H_{n-1}(F_\theta, \mathbb{Z}) = \mathbb{Z}^k$, which by the theorem has a free component of rank μ. Therefore

$$\pi_{n-1}(F_\theta) \cong H_{n-1}(F_\theta, \mathbb{Z}) \cong \mathbb{Z}^\mu,$$

and

$$H_j(F_\theta; \mathbb{Z}) = 0 \quad \text{for } j < n-1.$$

With this we can completely describe the homology groups of the fiber F_θ:

$$H_j(F_\theta; \mathbb{Z}) = \begin{cases} \mathbb{Z}^\mu & \text{for } j = n-1 \\ 0 & \text{otherwise} \end{cases}$$

Poincaré duality (2.7) implies this also describes the cohomology groups (since F_θ is orientable, even parallelizable):

$$H^j(F_\theta; \mathbb{Z}) = \begin{cases} \mathbb{Z}^\mu & \text{for } j = n-1 \\ 0 & \text{otherwise} \end{cases}$$

References

[1] Brieskorn, E, Knörrer, H.: *Plane Algabreic Curves*, Birkhäuser, 1986

[2] Milnor, J.: *Singular Points of Complex Hypersurfaces*, Princeton Univerity Press, 1968

[3] Hatcher, A.: *Algebraic Topology*, Cambridge University Press, 2002