Projective algebraic plane curves

Introduction

This report provides a brief introduction to algebraic curves, a branch of algebraic geometry that combines ideas from linear algebra and abstract algebra. While linear algebra studies systems of linear equations and algebra focuses on polynomials, algebraic geometry investigates systems of polynomial equations in several variables over a field K.

Because polynomial equations arise naturally throughout mathematics and its applications, algebraic geometry has developed into a broad and influential field with connections to areas such as commutative algebra, number theory, topology, complex analysis, and cryptography.

This report focuses on one of the simplest nontrivial cases: a single polynomial equation in two variables. The set of its solutions can be viewed as a plane algebraic curve. Studying these curves allows us to explore geometric questions and answer them using algebraic methods, leading to many interesting and important results that we will explore here.

$\mathbb{A}^2_K := K^2$ denotes the affine plane over the field K. From here on, we will focus on the case where $K = \mathbb{C}$. For $f \in \mathbb{C}[x, y]$ the set $V(f) := \{(x, y) \in \mathbb{A}^2 \mid f(x, y) = 0\}$ is called the *zero set of f.* A subset $C \subset \mathbb{A}^2$ is called a *plane affine algebraic curve* if there exists a nonconstant polynomial $f \in \mathbb{C}[x, y]$ such that $C = V(f)$. To obtain beautiful global properties, one can complete affine curves into projective curves by adding "points at infinity" through homogenization of the polynomial f. A subset $C \subset \mathbb{CP}^2$ is called a *projective algebraic curve* if there exists a homogeneous polynomial $F \in \mathbb{C}[x, y, z]$ with $deg(F) > 0$ such that $C = \{P \in \mathbb{CP}^2 \mid F(P) = 0\}$.

1 Local properties

We want to investigate the local geometry of a curve; hence, our first goal is to understand the *intersection multiplicity* of two curves at one of their intersection points. To do this, one would introduce the notion of *local ring* of $\mathbb{A}^2$: $\mathbb{C}[[x, y]] = \{f = \sum\limits_{i,j \geq 0} a_{ij} x^i y^j\}$. It captures the local geometry of the plane around a point. It is a UFD, and it has exactly one maximal ideal: $m_{x,y} = (x, y)$, where $f \in m_{x,y} \Leftrightarrow a_{00} = 0$ and f is inverible $\Leftrightarrow f \in C[[x, y]] \setminus m_{x,y}$.

From (x, y) to $(x', y') = (ax + by + c, dx + ey + f)$ for $a, b, c, d, e, f \in \mathbb{C}$ with $ae - bd \neq 0$ is an invertible affine coordinate transformation. With this, we can simplify our calculations by picking suitable coordinates. For example, instead of an arbitrary point $P \in \mathbb{A}^2$, it suffices to reduce our calculations to the origin (O).

Definition 1.1. Let $f, g \in \mathbb{C}[[x, y]]$ with no common irreducible components. The *intersection multiplicity* of f and g at O is $i_O(f, g) = dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{(f,g)}$, where (f, g) is the ideal generated by f and g in $\mathbb{C}[[x, y]]$.

Remark 1.2. In general $1 \leq i_O < \infty$. ($i_O = \infty$ if f, g have a common irreducible component.)

Definition 1.3. Let P be a point on the curve C and let T_pC be the tangent line at P. (Hence, $i_P(C, T_PC) \geq 2$.) If $i_P(C, T_PC) \geq 3$ we call P *inflection point*.

Let $C = \{f(x, y) = 0\} \subset \mathbb{A}^2$. We call the point $P \in C$ *smooth* if and only if $(\frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P)) \neq (0, 0)$; otherwise, P is *singular*.
Let $f(P) \overset{Taylor}{=} f_0 + f_1(P) + f_2(P) + \dots$ be the Taylor expansion of f at P. We define the *multiplicity of f at point P* the integer $m := m_P(f) \in \mathbb{N}$, which satisfies $f_m(P) \neq 0$ and $f_i(P) = 0$ for all $i < m$. Note that $m = 1 \Leftrightarrow (0, 0)$ is smooth point.

Properties of i_O.

1. $i_O(g, f_1 \cdot f_2) = i_O(g, f_1) \cdot i_O(g, f_2)$;
2. $i_O(f, g) \geq m_O(f) \cdot m_O(g)$, with equality occurring if and only if f and g have no tangent 'lines' (lines in the tangent cone) in common at O.

2 Bézout theorem

The projective plane was constructed so that any two distinct lines intersect at exactly one point. Bézout's famous theorem shows that a much stronger statement holds.

Theorem 2.1. ***(Bézout)***
Let $C, D \subset \mathbb{CP}^2$ be projective curves with no common components, and set $deg(C) = c,\ deg(D) = d$.
$C = \{f(x, y, z) = 0 \mid f$ is a homogen ploynomial, $deg(f) = c\}$,
$D = \{g(x, y, z) = 0 \mid g$ is a homogen ploynomial, $deg(g) = d\}$.
Then $\sum_{P \in C \cap D} i_P(f, g) = c \cdot d$.

Corollary 2.2. *If C and D meet in $c \cdot d$ distinct points, then these points are all smooth points on both C and D and the intersection is transversal.*

Corollary 2.3. *If two curves of degrees c and d have more than $c \cdot d$ points in common, then they have a common component.*

The 2.1 theorem is a simple yet powerful result that is widely used in many proofs. To mention just a few examples:

1. All smooth complex projective curves of degree 2 (*conic*) are topologically (even algebraically) equivalent to the Riemann sphere ($\mathbb{CP}^1$).

2. A complex projective curve of degree 3 (*cubic*) in $\mathbb{CP}^2$ which has more than one singular point must be reducible.

3. Let C and D be complex projective curves of degree 3 and $C \cap D = \{P_1, \ldots, P_9\}$. If P_1, P_2, P_3 are on a line L then there is a curve Q with degree 2, such that $P_4, P_5, \ldots, P_9 \in Q$. And if $P_1, P_2, \ldots P_6$ are on a curve Q with $deg(Q) = 2$ then there is a line L such that $P_1, P_2, P_3 \in L$.

4. **First Splitting Theorem**
   Let $C, D \subset \mathbb{CP}^2$ be projective curves without common components, such that $deg(C) = deg(D) = n \in \mathbb{N}$. Let $C \cap D = \{P_1, P_2, \ldots, P_{n^2}\}$. Assume that there exists a projective curve $E \subset \mathbb{CP}^2$ such that E is irreducible, $deg(E) = e \in \mathbb{N}$, $e < n$ and $P_1, P_2, \ldots, P_{ne} \in E$. Then there exists $F \subset \mathbb{CP}^2$ of degree $n - e$ such that $P_{ne+1}, \ldots, P_{n^2} \in F$.

5. **Pappus' theorem** Let L_1, L_2 be two lines; $P_1, P_2, P_3 \in L_1$, $Q_1, Q_2, Q_3 \in L_2$ none of these points are in in the intersection of the two lines. Let L_{ij} be the line between P_i and Q_j. For each i, j, k with $\{i, j, k\} = \{1, 2, 3\}$, let $R_k = L_{ij} \cap L_{ji}$. Then R_1, R_2, R_3 are collinear.

6. **Pascal's theorem** Let C be an irreducible projective conic. Pick six distinct points $P_1, \ldots, P_6$ on C (that can be thought of as the vertices of a hexagon inscribed in C). Then the intersection points of the opposite edges of the hexagon lie on a line.

7. **Second Splitting Theorem**
   Let $C, D \subset \mathbb{CP}^2$ be projective curves without common components, such that $deg(C) = n$ and $deg(D) = m$. Let $C \cap D = \{P_1, P_2, \ldots, P_{nm}\}$. Assume that there exists a line $L \subset \mathbb{CP}^2$ such that $P_1, P_2, \ldots, P_n \in L$. Then there exists $E \subset \mathbb{CP}^2$ of degree $m - 1$ such that $P_{n+1}, \ldots, P_{nm} \in E$.

3 Group structure of a smooth curves of degree 3

The group structure on an elliptic curve (e.g. on a smooth plane curve of degree 3) is significant because it connects geometry with algebra in a deep and powerful way. Points on such curves can be “added” together according to a geometric rule, turning the set of points into an algebraic group.

One can show that if a smooth manifold X admits a smooth group structure, then its Euler characteristic must be zero. Hence, in the smooth curve case, its genus must be 1. Applying the genus formula for plane curves, $2g = (d-1)(d-2)$, we conclude that the degree of the curve must be 3. Therefore, in this chapter the curve C is smooth with degree 3.

Let a point O be fixed on the curve C. Define the group operation $\oplus : C \times C \to C$ by assigning to each pair of points (A, B) the point $A \oplus B$. Let L_{AB} denote the line passing through A and B. If $A = B$, then $L_{AB} = T_A C$. This line intersects the curve at a third point, denoted by P. Next, consider the line L_{OP} passing through O and P. The third intersection point of L_{OP} with the curve is defined to be the point $A \oplus B$.

It is easy to verify that O serves as the identity element, that every point admits an inverse, and that the operation is commutative. The most nontrivial property to establish is associativity. However, by applying the First Splitting Theorem, one can show that the operation is associative as well.

Once $O \in C$ is fixed, let $K \in C$ be defined 'divisorially' as $T_O C \cap C = O + O + K$.

Theorem 3.1. *Let D be a curve of degree d. $C \not\subset D$ and $C \cap D = \{P_1, \ldots, P_{3d}\}$. Then $P_1 \oplus P_2 \oplus \cdots \oplus P_{3d} = K^{\oplus d}$.*

Corollary 3.2. *With the notation of the previous Theorem, assume that D is of degree $deg(D) = 3$. Let E be another curve of degree 3 such that $P_1, \ldots, P_8 \in E$. Then $P_9 \in E$ too.*

Remark 3.3. If the tangent line $T_O C$ intersects the curve C at the points $O, O,$ and K, then $i_O(C, T_O C) \geq 2$. If O is an inflection point, then $i_O(C, T_O C) = 3$ hence, the third intersection point coincides with O, i.e. $K = O$.

If $K \in C$ is constructed as in the previous remark (that is, O is chosen as an inflection point, hence $K = O$) then the following statements are equivalent for the points $A_1, A_2, A_3 \in C$:

(a) There is a line L, such that $L \cap C = \{A_1, A_2, A_3\}$.

(b) In $(C, O, \oplus)$: $A_1 \oplus A_2 \oplus A_3 = O$.

Corollary 3.4. *Suppose O is an inflection point of C. The point $I \in C$ is an inflection point if and only if $I^{\oplus 3} = O$.*

Corollary 3.5. *The set of inflection points forms a subgroup of $(C, O, \oplus)$ isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$.*

4 Divisors

In this chapter, we continue our study of smooth projective curves over an algebraically closed field, which in our setting is the field $\mathbb{C}$. The curve $C \subset \mathbb{CP}^2$ will be assumed to have degree d, and it is smooth. Divisors provide a very convenient piece of notation that allows us to consider multiplicities at all points of a curve simultaneously.

Definition 4.1. *Divisor* on C is a free $\mathbb{Z}$-module generated by points on C. It is denoted by $Div(C) = \mathbb{Z}\langle P \rangle_{P \in C} = \{\sum_{finite\ sum} n_i P_i \mid n_i \in \mathbb{Z},\ P_i \in C\}$.

Definition 4.2. The *degree of a divisor* $\sum_i n_i P_i$ is the number $\sum_i n_i \in \mathbb{Z}$. The degree map is a group homomorphism, $deg\colon Div(C) \to \mathbb{Z}$. It is surjective. Its kernel is denoted by $Div^0(C) = \{D \in Div(C) \mid deg(D) = 0\}$.

Definition 4.3. A divisor on C is called *principal* if it is the divisor of a non-zero rational function. The group of principal divisors is denoted with $Prin(C)$.

$S(C) := \mathbb{C}[x, y, z]/(C)$ is the *homogeneous coordinate ring* of C. A non-zero element $f \in S(C)$ is called homogeneous of degree d if it can be represented by a homogeneous polynomial of degree d in $\mathbb{C}[x, y, z]$. The vector space of these elements, together with 0, will be denoted $S_d(C)$. The *rational functions* on C is defined as $K(C) := \{\frac{f}{g} \mid f, g \in S_d(C)$ for some $d \in \mathbb{N}\}$.

The divisor of a rational function $f = \frac{P}{Q} \in K(C)$ is $div(\frac{P}{Q})$ = roots of f − poles of f along C=

$$\sum_{p\in C\cap\{P=0\}} i_p(C, P) - \sum_{q\in C\cap\{Q=0\}} i_q(C, Q).$$

Therefore, $Prin(C) = \{div(f) \mid f \in K^*(C)\}$, where div is a group homomorphism between groups $(K^*(C), \cdot)$ and $(Div(C), +)$. So, $Prin(C) = Im(div) = div(K^*(C)) < Div(C)$.

Remark 4.4. If $f \in K^*(C)$ then $deg(div(f)) = 0$. In particular, $Prin(C) < Div^0(C)$.

Definition 4.5. The quotient group $DivCl(C) := \frac{Div(C)}{Prin(C)}$ is called the *divisor class group of C*. The morphism deg induces a surjective morphism $deg : DivCl(C) \to \mathbb{Z}$ whose kernel is $DivCl^0(C) := \frac{Div^0(C)}{Prin(C)}$.

Definition 4.6. Two divisors D_1 and D_2 defining the same element in $DivCl(C)$, i. e. with $D_1 - D_2 = div(f)$ for a rational function $f \in K^*(C)$, are said to be *linearly equivalent*, written $D_1 \sim D_2$.

Theorem 4.7. *If C is smooth of degree $d \leq 2$ then $DivCl^0(C) = 0$.*

However, for $d = 3$ we have the following statement.

Theorem 4.8. *Assume that C is smooth of degree 3. Fix $O \in C$. Then there is a group isomorphism between the divisor class group of C of divisors of degree zero and the group on C from the previous chapter: $(DivCl^0(C), +) \xrightarrow{\sim} (C, O, \oplus)$.*

References

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