ELTE MATH MSC, DIRECTED STUDIES II

Geometric representations of planar graphs and antiplanar graphs

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For this semester's Directed Studies my goal was to familiarize myself with the geometric results, techniques, methods of proofs which can be used for researching the properties of antiplanar graphs which I also got introduced to this semester by my advisor, Gábor Damásdi. The family of antipőlanar graphs hgas many equivalent definitions, for example the edge graphs of 3-dimensional graphs that only have edges of length 1.
I am reading selected chapters from the book Graphs and Geometry from László Lovász. The chapters that I read are in order Chapter 2 Planar Graphs, Chapter 3 Rubber Bands, Chapter 5 Coin Representation, Chapter 14 Stresses and I have started reading Chapter 15 Rigidity and Motions of Frameworks.

1 PLANAR GRAPHS

This is an introductory chapter with very few proofs, mostly stating results that I previously learned in other geometric or combinatorial classes. The chapter starts by definitions connected to planar graphs and introducing the notion of duality. It continues with Euler's formula ($n - e + f = 2$ for connected planmar graphs) and some of its corollaries, for example that every planar greaph has a node with degree at most 5 and every triangle-free planar graph has a node with degree at most 3. It next states Kuratowski's theorem, which says that a graph can be embedded into the euclidean plane if and only if it has no subgraph homeomorphic to the complete bipartite graph $K_{3,3}$ or to the complete graph K_5.
The next result is a lemma which will be used for example in the proof of Cauchy's theorem of the stresses of 3-polytopes. The lemma states that every 2-colored planar graph has two

quiet nodes, which are nodes such that its edges of one color are consecutive in the given geometric representations. The proof of the lemma is a double counting of bicolored corners. Every non-quiet node has at least 4 bicolored corners and counting the faces every $2r$-face and every $2r+1$-face has at most $2r$ bicolored corners. The result of the lemma then follows from Euler's formula. The section of the book finishes with planarity results regarding 3-connected graphs which are useful for analyzing graphs of 3-dimensional polytopes, because they are 3-connected.

The chapter finishes with stating results relatod to straight line representations. It states that every planar graph has an embedding with straight lines. It then states that the graph of every 3-polytope can be drawn in thje plane and its converse, Steinitz's theorem which states that a planar graph is isomorphic to a skeleton of a 3-polytope if and only if it is 3-connected. The chapter finishes with introducing Voronoi and Delauney diagrams.

2 RUBBER BANDS

The chapter starts with an informal and then a formal definition of rubber band representations. The informal description is as follows, think of some nonempty subset of the nodes to be nailed to their positions and the edges connecting nodes (containing the non-nailed nodes) are replaced by rubber bands satisfying Hooke's Law, which means that the force to extend a band by some distance scales linearly with the displacement and the rubber bands form a straight-line representation. This means that the energy of the rubber bands are quadratic functions of the displacements. And a rubber band representation is the equilibrium position of tha nodes which we later prove in the chapter that it is unique by proving that the energy function is strictly convex and usiong that the equilibrium position is the minimum of the energy. The next result is Tutte's result which states that every rubber band representation of a simple 3-connected planar graph gives a straight line representation of it, moreover, every country is a convex polygon. The chapter finishes with a Linial-Lovász-Wigderson result and its later improvements about how rubber band representations can be used to test the k-connectivity of graphs.

3 COIN REPRESENTATION

One of the deepest result of representatiopns of planar graphs is Koebe's coin representation theorem from 1963. It states that every planar graph can be represented as the tangency graph of a nonoverlapping family of circular disks. The tangency graph is connecting the cenetrs of tangent circles with a straight line. It means that the existence of a straight line representation is a corollary of this result. zThje representation is essentially unique, meaning up to inversions to a different circle, if the graoh is a triangulation. These inversions mean that every coin representation can be turned "inside out", every face of the graph is the outer face of a coin representation.

After the proof of the theorem, the chapter disusses corollaries of the result, for example formulations in space, one of these is every 3-connected planar map G has a proper double cap representation on the sphere. Another one is the cage theorem which states that every 3-connected planar graph can be represented with a 3-polytope with edges tangent to the same given sphere. The following section discusses the connection with the Riemann mapping theorem from complex analysis. The chapter finishes with further applications, planar separators, Laplacians of planar graphs and general convex bodeis and domains.

4 STRESSES

This chapter is the continuation of Chapter 3 (Rubber Bands). We can think of graphs as bar-and-joint frameworks. If we replace one bar with a rubber band (trying to contract), the structure will either move or carry some stress. We can extend the model by thinking of some edges as "cables" that can only carry positive stress and "struts" which can only carry negative stress. The chapter starts by formal definitions and linear algebraic results, followed by stresse of convex polygons. In, 3 dimensions, Cauchy's classic result is that the greaph of a convex 3-polytope cannot carry nonzero stresses. Its proof uses the lemma of quiet nodes from the Planarf graph chapter after leaving edges with zero stress and 2-coloring based on the sign of the stress. A quiet node then cannot exist because it would move because there is a direction which every stress pushes or pulls it. The chapter then finishes by discussing results related to braced stresses which means that in addition to the edges, every node is connected to the origin.