

Stable packing of planar convex bodies

Sándor Fazekas

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1 Introduction

In this paper, we will talk about convex bodies on the plane with positive measure. Two convex bodies touch if their boundaries have non-empty intersection. They intersect if their interiors have non empty intersection.

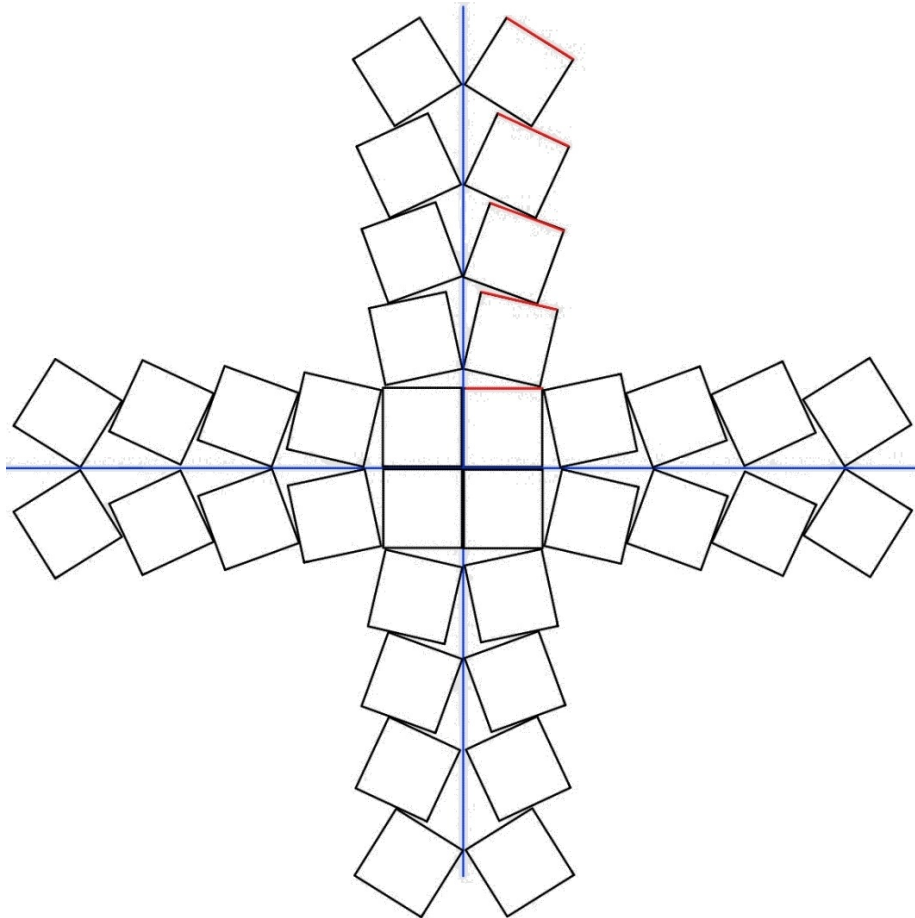
Definition 0.1. (Translation stable packing (Böröczky packing))

A packing of a family of convex bodies on the Euclidean plane is *translation stable*, if for every member of the family K and every translation $t \in \mathbb{R}^2$, there is a scalar $s \in (0, 1]$ for which $K + st$ intersects another member of the family.

Böröczky showed that there is a stable packing of circles of zero density [1]. We will present examples of convex bodies that have a translation stable packing of zero density. First, we will show the construction for the square and apply it to other convex bodies

Theorem 1. There is a translation stable packing of congruent squares of zero density.

Proof. First, let S_1 be $[0, 1]^2$ on the plane. Let e_1 and e_2 denote the standard basis $(1, 0)$ and $(0, 1)$. Place another square on top of S_1 and call it S_2 . Rotate S_2 by $\varepsilon/2$ degrees around the vertex of S_1 $(1, 1)$. After the rotation, translate S_2 along the top side of S_1 , such that S_2 touches the y -axis and S_1 . Now we do the same procedure again relative to S_2 (by which we mean that the basis rotates by the angle S_2 is rotated), however, the rotation halves after every step. In this way, no square will be rotated more than ε degrees relative to S_1 . Finally, reflect them to the y -axis and rotate it three times by 90 degrees. If some of the squares overlap, we consider them the same.

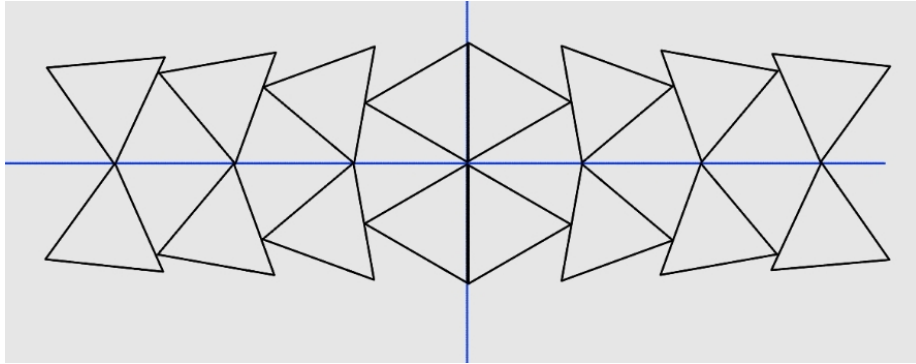


To prove that the packing is stable, we have to check that no square can be translated. We will prove that S_i cannot be moved for any $i \in \mathbb{N}$. S_i is adjacent to S_{i+1} , S_{i-1} and its reflection, we will call it R_i . Let $v \in \mathbb{S}^1$ be a unit vector, we define $v^+ = \{x \in \mathbb{S}^1 \mid \langle x, v \rangle > 0\}$. Let $r^+(\varepsilon)$ denote the rotation by ε degrees, similarly $r^-(\varepsilon)$ is the rotation by $-\varepsilon$ degrees. S_{i+1} and S_{i-1} , blocks S_i to move in the directions $(-r^-(\varepsilon/2^{i+1})e_2)^+ \cup (r^-(\varepsilon/2^{i+2})e_2)^+$. Also, R_i blocks S_i to move in directions $r^+((90 - \varepsilon/2^{i+1})e_2)^+ \cap r^+((90 + \varepsilon/2^{i+1})e_2)^+$. These directions cover the hole \mathbb{S}^1 . For S_1 it is easy to see that it is stable, since it has contact points with other members on all four sides. This proves the theorem. \square

Theorem 2. There is a translation stable packing of congruent regular triangles of zero density.

Proof. This packing has the same setup as the previous one. We have a regular triangle T_1 in the first quadrant, with height 1 and an edge on the x -axis.

T_2 will be the translation of T_1 by e_2 , rotated by $\varepsilon/2$ degrees around of the touching point of T_1 and T_2 and translated by a multiple of $r^-(\varepsilon/2)e_1$ such that it touches the y -axis and T_1 . This is repeated relative to T_i with rotation $\varepsilon/2^i$ degrees. The difference here is we finish by reflecting the triangles to the x -axis and a final reflection to the y -axis gives us a translation stable packing of regular triangles.



□

Theorem 3. There is no finite family of convex sets for which a stable packing can be achieved.

Proof. First, we assign a higher-lower relation for every touching pair of convex bodies. Let e_1, e_2 be an orthonormal basis in \mathbb{R}^2 . We say that a convex body K_1 is higher than a convex body K_2 , if they touch each other and there are scalars $s \in (0, 1]$, $c \in (0, s]$, such that $K_1 + ce_2$ does not intersect K_2 for every c . Obviously, this is not a total ordering, some members may not be related, and some members can be both above and under each other.

For a packing to be stable, there has to be a member above every member. We can make a directed graph from the convex bodies and the relations amongst them. Every convex body K_i will be represented as a vertex V_i and every relation as a directed edge. If K_1 is higher than K_2 , it is represented as an edge pointing from V_2 to V_1 . If two members are not related, then there is no edge between them. If there is a finite packing, then in the higher-lower directed graph there has to be a cycle since no member can be moved in the direction e_2 . This means that from every vertex there is an edge that points to a higher vertex. The graph is finite, so there has to be at least one cycle. We will show that it is not possible.

Definition 3.1. (Skirt)

The skirt of a convex body C on the plane is defined by the following properties:

$$S(C) = \{x \in \mathbb{R}^2 \mid x = y - se_2, s \in [0, \infty), y \in C\}.$$

Also the upper skirt is defined the following way:

$$S^+(C) = \{x \in \mathbb{R}^2 \mid x = y + se_2, s \in [0, \infty), y \in C\}.$$

Assume that there is a family of convex bodies $K_1, \dots, K_n \subset \mathbb{R}^2$ packed on the plane such that K_{j+1} is higher than K_j for every $0 < j < n$. It is easy to see that K_{i+1} is higher than K_i if they touch each other, and $K_{i+1} \cap \text{int}(S^+(K_i)) \neq \emptyset$. If it forms a cycle in the graph defined above, that would mean that K_n intersects $S(K_1)$. However, K_n is higher than K_{n-1} so $K_n \cap \text{int}(S^+(K_{n-1})) \neq \emptyset$. Since K_n is convex, it cannot intersect $\text{int}(S(K_{n-1}))$, because the line segment connecting a point of $\text{int}(S(K_{n-1}))$ to a point of $\text{int}(S^+(K_{n-1}))$ would intersect K_{n-1} . For the same reasons K_n cannot intersect $\text{int}(S^+(K_1))$. K_n cannot intersect the set $\text{int}(S^+(K_1)) \cup \text{int}(S(K_{n-1})) \cup \bigcup_{i=1}^{n-1} K_i$, which cuts the plane into two regions. However, K_n has to intersect $\text{int}(S^+(K_{n-1}))$ and $\text{int}(S(K_1))$, but they are in different regions. This means that there is no convex body which can satisfy these conditions.

□

References

- [1] K. Böröczky. “Über stabile Kreis- und Kugelsysteme”. German. In: *Ann. Univ. Sci. Budap. Rolando Eötvös, Sect. Math.* 7 (1964), pp. 79–82. ISSN: 0524-9007.