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Hodge theory

Supervisor:
Árpád Tóth
Professor

Author:
Fogarasi András
Mathematics MsC student

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Introduction

Categorising different objects is one of the basic problems in geometry. In this report, we explore one specific family of objects: Hermitian manifolds. These may be viewed as Riemann manifolds with some extra complex structure. As such, they have some stronger properties than general Riemann manifolds. They include, most importantly, non-singular complex varieties. As such, their study is quite useful in that field.

In Section 1, we will introduce some analytic tools on complex manifolds. This is necessary to understand the remainder of the material.

In Section 2, we describe the proof of the Hodge theorem, which establishes a correspondence between the Dolbeault cohomology groups and harmonic differential forms; those that are in the kernel of a certain elliptic operator, the $\bar{\partial}$-Laplacian. These results are applicable not only to the complex case. Such a correspondence exists even in the real case, with the de Rham cohomology and the Hodge Laplacian.

In Section 3, we describe two geometric identities that arise as a result of Hodge theory. These are Serre duality – describing an automorphism of the Dolbeault cohomology – and the Künneth theorem, through which we can determine the cohomology of a product space.

Finally, in Section 4, we introduce a special type of complex manifold: The Kähler manifold. These may be viewed as a sort of "locally Euclidean" Hermitian manifold. They occur, for example, as projective complex varieties. Through Hodge theory, we can prove a number of symmetries. We will first describe a decomposition of the de Rham cohomology into a direct sum involving the Dolbeault cohomology; which cannot generally be done in a more general case. We also discuss an application of representation theory, which allows us to define the so-called Lefschetz decomposition of the de Rham cohomology.

1 Calculus on complex manifolds

1.1 Hermitian metrics

We wish to do calculus on complex manifolds. For some $E \to M$ vector bundle, suppose that we have positive-definite Hermitian inner products $h_x \in E_x \otimes \overline{E_x} \to \mathbb{C}$ such that the scalar fields $h_{ij}(x) = h_x(\xi_i(x), \overline{\xi_j(x)})$ are smooth. Let $\{\xi_i\}_i$ be a unitary frame, $\{\varphi_i\}_i$ its dual. The Hermitian metric may then be expressed as

$$ds^2 = \sum_j \varphi_j \otimes \overline{\varphi_j}$$

where φ_i is the dual of ξ_i. The point-wise inner product of two sections is also denoted by $\langle \alpha, \beta \rangle$. A complex manifold with a Hermitian metric is called a Hermitian manifold.

We examine more closely the Hermitian metrics over the bundle of holomorphic functions on M, which we denote by $T'M$. Each φ_i is now a differential form of type $(1,0)$. We call the metric's associated $(1,1)$-form

$$\omega = \frac{i}{2} \sum_j \varphi_j \wedge \overline{\varphi_j}$$

The choice of the normalising factor is motivated by the fact that if we decompose into real parts as $\varphi_i = \alpha_i + i\beta_i$, then this allows us to write ω equivalently as

$$\begin{aligned} \omega &= \frac{i}{2} \sum_j \varphi_j \wedge \overline{\varphi_j} \\ &= \sum_j \alpha_j \wedge \beta_j \end{aligned}$$

This gives way to the so-called Wirtinger theorem. A Hermitian metric on $\mathbb{C}^n$ naturally induces a Riemann metric on $\mathbb{R}^{2n}$. This is

$$\mathrm{Re}(ds^2) = \sum_j \alpha_j \otimes \alpha_j + \beta_j \otimes \beta_j$$

The volume form induced by this is $d\mu = \alpha_1 \wedge \beta_1 \wedge \cdots \wedge \alpha_n \wedge \beta_n$. With this choice of normalising factor for ω, we have the Wirtinger theorem:
Theorem (Wirtinger)

$$d\mu = \frac{\omega^n}{n!}$$

This is made more novel by the action of a pullback on the associated form. Let $f : N \to M$ be a holomorphic map such that $f_* : T'_z(N) \to T'_{f(z)}(M)$ is injective for all $z \in N$. This defines the pullback metric on N defined by

$$\left\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right\rangle_z = \left\langle f_* \frac{\partial}{\partial z_i}, f_* \frac{\partial}{\partial z_j} \right\rangle_{f(z)}$$

The associated $(1,1)$-form of N with the pullback metric is just the pullback form. If N and M are of dimensions n and m, then we may choose a coframe such that $f^*\varphi_j = 0$ for $j \geq n+1$. Then

$$\begin{aligned}\omega_N &= \frac{i}{2}\sum_{j=1}^{n} f^*\varphi_j \wedge f^*\overline{\varphi_j} \\ &= f^*\left(\frac{i}{2}\sum_{j=1}^{n} \varphi_j \wedge \overline{\varphi_j}\right) \\ &= f^*\left(\frac{i}{2}\sum_{j=1}^{m} \varphi_j \wedge \overline{\varphi_j}\right) \\ &= f^*\omega_M\end{aligned}$$

This is one interesting structure that Hermitian manifolds have. For example, a curve in $\mathbb{R}^2$ has the arc length as its volume form. If Wirtinger-like theorem were true for Riemann manifolds, we would expect that all such volume forms are a pullback of some global form on $\mathbb{R}^2$. This is not the case.

It is also due to this pullback property that we can say that every projective complex variety is "Kähler", a concept that will be introduced in Section 4. However, projective varieties will not be discussed.

1.2 Connections on complex manifolds

Definition (Connections)

A connection on a complex vector bundle $E \to M$ is a linear map

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$$

from smooth sections of E to smooth sections of $T^*M \otimes E$, satisfying the Leibniz rule

$$\nabla(f \cdot \zeta) = df \otimes \zeta + f \cdot \nabla\zeta$$

for all $f \in C^\infty(M, \mathbb{R})$, $\xi \in \Gamma(E)$.

Connections generalise the derivative in a global fashion, making it a useful tool in analysis on line bundles.
A connection on E automatically defines connections on $(T^*M)^{\wedge k} \otimes E$. We require that

$$\nabla(\psi \otimes s) = d\psi \otimes s + (-1)^k \psi \wedge \nabla s$$

when $\psi \in \Gamma((T^*M)^{\wedge k})$ and $s \in \Gamma(E)$. Most importantly, this definition allows us to apply the connection multiple times in succession, since it sends $\Gamma((T^*M)^{\wedge k} \otimes E)$ to $\Gamma((T^*M)^{\wedge k+1} \otimes E)$. This extension is called the covariant exterior derivative induced by a connection.
An arbitrary connection is not a particularly nice object, though it is immensely useful. But when M is complex and E Hermitian, we have a natural choice connection with two defining properties.

1. Using the decomposition $T^*M = T^{*\prime} \oplus T^{*\prime\prime}$ on complex manifolds, one can project a connection into these spaces, writing $\nabla = \nabla' + \nabla''$ where

$$\begin{aligned}\nabla' &: \Gamma(E) \to \Gamma(T^{*\prime} \otimes E) \\ \nabla'' &: \Gamma(E) \to \Gamma(T^{*\prime\prime} \otimes E)\end{aligned}$$

We say that the connection is compatible with the complex structure when $\nabla'' = \bar{\partial}$.

2. Since E is Hermitian, we have have an inner product $\langle .,. \rangle$. We say that a connection is compatible with this metric when

$$d\langle \xi, \eta \rangle = \langle \nabla \xi, \eta \rangle + \langle \xi, \nabla \eta \rangle$$

There is a unique connection ∇ satisfying these two properties, called the **Chern connection**.

Locally, given a holomorphic frame $\{e_i\}_i$, let $h_{i\bar{j}} = \langle e_i, e_j \rangle$ be a scalar matrix and $h^{i\bar{j}}$ its inverse. The Chern connection is defined by

$$\nabla_{e_i} e_j = \sum_{k,l} (h^{k\bar{l}} \partial_i h_{j\bar{l}}) e_k$$

since then

$$\begin{aligned}
\langle \nabla e_i, e_j \rangle &= \sum_{k,l} (h^{k\bar{l}} \partial h_{i\bar{l}}) h_{k\bar{j}} = \partial h_{i\bar{j}} \\
\langle e_i, \nabla e_j \rangle &= \overline{\sum_{k,l} (h^{k\bar{l}} \partial h_{j\bar{l}}) h_{k\bar{i}}} = \bar{\partial} h_{i\bar{j}} \\
d\langle e_i, e_j \rangle &= \partial h_{i\bar{j}} + \bar{\partial} h_{i\bar{j}} = \langle \nabla e_i, e_j \rangle + \langle e_i, \nabla e_j \rangle
\end{aligned}$$

It is compatible with the complex structure since each e_i is holomorphic and each $\sum_k h^{i\bar{k}} \partial h_{j\bar{k}}$ is of type $(1,0)$. From the first, we have $\bar{\partial} e_i = 0$. From the second, we have $\nabla'' e_i = 0$. The definition turns out to be coordinate-independent, so we get a globally defined connection.

1.3 Functional analysis

A Hermitian metric allows us to define inner products on the tensor spaces $A^{p,q}(M)$ of smooth (p,q)-type differential forms. Locally, we choose the basis $\{\varphi_I \wedge \overline{\varphi}_J\}_{|I|=p,|J|=q}$ to be orthogonal, with $\|\varphi_I \wedge \overline{\varphi}_J\| = 2^{p+q}$. The normalisation is again motivated by wanting to align with the induced real metric.

This now allows us to define global inner products of forms:

$$\langle \psi, \eta \rangle = \int_M \langle \psi, \eta \rangle d\mu$$

This turns $A^{p,q}(M)$ into a pre-Hilbert space. We define the Hodge star operator $* : A^{p,q} \to A^{n-p,n-q}$ such that

$$\langle \psi, \eta \rangle d\mu = \psi \wedge *\eta$$

Locally, it is given by

$$\begin{aligned}
\eta &= \sum_{I,J} \eta_{I\bar{J}} \varphi_I \wedge \overline{\varphi}_J \\
*\eta &= 2^{p+q-n} \sum_{I,J} \varepsilon_{I\bar{J}} \overline{\eta_{I\bar{J}}} \varphi_{I_0} \wedge \overline{\varphi}_{J_0}
\end{aligned}$$

where I_0 and J_0 are just the complements of I and J, and ε is a sign. The signs work out such that $**\eta = (-1)^{p+q}\eta$ We wish to define an adjoint operator $\bar{\partial}^* : A^{p,q} \to A^{p,q-1}$ such that $\langle \bar{\partial}^*\psi, \eta\rangle = \langle \psi, \bar{\partial}\eta\rangle$. As it turns out, $\bar{\partial}^* = -*\bar{\partial}*$ works. Letting $\eta \in A^{p,q}$ and $\psi \in A^{p,q-1}$, we have:

$$\begin{aligned}\langle \bar{\partial}\psi, \eta\rangle &= \int_M \bar{\partial}\psi \wedge *\eta \\ &= \int_M \bar{\partial}(\psi \wedge *\eta) + (-1)^{p+q} \int_M \psi \wedge \bar{\partial}*\eta\end{aligned}$$

Here $(\psi \wedge *\eta)$ is an $(n, n-1)$-form, so $\bar{\partial}(\psi \wedge *\eta) = d(\psi \wedge *\eta)$. By Stokes's theorem, its integral is 0. As such:

$$\begin{aligned}\langle \bar{\partial}\psi, \eta\rangle &= (-1)^{p+q} \int_M \psi \wedge \bar{\partial}*\eta \\ &= \int_M \psi \wedge *(-*\bar{\partial}*\eta) \\ &= \langle \psi, \bar{\partial}^*\eta\rangle\end{aligned}$$

With the adjoint operator, we can define the $\bar{\partial}$-Laplacian, or the Dolbeault Laplacian.

$$\begin{aligned}\Delta_{\bar{\partial}} &= \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^* \\ &= (\bar{\partial} + \bar{\partial}^*)^2\end{aligned}$$

We will usually denote it by just Δ. We call a form harmonic if its Laplacian vanishes. We denote the space of harmonic forms by $\mathcal{H}^{p,q} = \mathrm{Ker}(\Delta) \cap A^{p,q}$.

Let us examine the quadratic form induced by Δ:

$$\begin{aligned}\langle \Delta\eta, \eta\rangle &= \langle \bar{\partial}^*\bar{\partial}\eta, \eta\rangle + \langle \bar{\partial}\bar{\partial}^*\eta, \eta\rangle \\ &= \|\bar{\partial}\eta\|^2 + \|\bar{\partial}^*\eta\|^2 \geq 0\end{aligned}$$

This tells us two things. Firstly, that Δ is positive semi-definite. And secondly, that η is harmonic if and only if it is $\bar{\partial}\eta = \bar{\partial}^*\eta = 0$. This will be useful for examining the minimum-norm problem later on.

As an aside, we can similarly define the adjoint of any linear operator. This allows us to define the Hodge Laplacian, also called the de Rham operator:

$$\begin{aligned}\Delta_d &= d^*d + dd^* \\ &= (d + d^*)^2\end{aligned}$$

Normally, $\Delta_{\bar{\partial}}$ and Δ_d are quite different. Later though, we will see that on a so-called Kähler manifold, $\Delta_d = 2\Delta_{\bar{\partial}}$

2 Hodge theorem

The Dolbeault cohomology group is not immediately easy to comprehend. It would be nice if we had a "natural" representative form of each class. One idea is to define a norm and try to choose the minimum norm element in each class. The only possible issue is that the space of differential forms is not complete; a minimum must not exist in general. Here, it will turn out that it does exist. These representatives will be precisely the harmonic forms.

2.1 The minimum norm problem

Given a $\bar{\partial}$-closed form η, which element has the smallest norm in the affine space $\eta + \bar{\partial}A^{p,q-1}$? We examine

$$\begin{aligned} \|\eta + \bar{\partial}\psi\|^2 &= \|\eta\|^2 + \|\psi\|^2 + \mathrm{Re}\langle \eta, \bar{\partial}\psi\rangle \\ &= \|\eta\|^2 + \|\psi\|^2 + \mathrm{Re}\langle \bar{\partial}^*\eta, \psi\rangle \\ &\geq \|\eta\|^2 + \mathrm{Re}\langle \bar{\partial}^*\eta, \psi\rangle \end{aligned}$$

and notice that if $\bar{\partial}^*\eta = 0$, then it is minimal. Conversely, if η is minimal, then it is a stationary point of the function $\|.\|^2$. But then:

$$\begin{aligned} \frac{\partial}{\partial t}\|\eta + t\bar{\partial}\psi\|^2 &= 2\mathrm{Re}\langle \eta, \bar{\partial}\psi\rangle \\ \frac{\partial}{\partial t}\|\eta + t\bar{\partial}(i\psi)\|^2 &= 2\mathrm{Im}\langle \eta, \bar{\partial}\psi\rangle \end{aligned}$$

So $\bar{\partial}^*\eta$ is actually orthogonal to all of $A^{p,q-1}$, which means it is 0.
If it were true that this minimum always exists, then this would imply that every Dolbeault cohomology class $H^{p,q}_{\bar{\partial}}$ has a unique representation in $\mathcal{H}^{p,q}$. In fact, it would imply that the two spaces are isomorphic. This is what we will show.

With the Chern connection ∇, we may define the Sobolev s-norm by

$$\|\alpha\|_s^2 = \sum_{k\leq s}\int_M \|\nabla^k\alpha\|^2 d\mu$$

The Sobolev spaces $\mathbf{H}_s^{p,q}$ are the completions of $A^{p,q}$ in these norms. Clearly, for $s \leq r$, $\mathbf{H}_s \subset \mathbf{H}_r$. What is less obvious is something we will not prove:
Lemma (Rellich)
The inclusion $\mathbf{H}_s \subset \mathbf{H}_r$ is compact.

We also define the Dirichlet inner product, along with its induced norm:

$$\begin{aligned} \langle \alpha, \beta\rangle_D &= \langle \alpha, (I+\Delta)\beta\rangle \\ &= \langle \alpha, \beta\rangle + \langle \bar{\partial}\alpha, \bar{\partial}\beta\rangle + \langle \bar{\partial}^*\alpha, \bar{\partial}^*\beta\rangle \\ \|\varphi\|_D^2 &= \langle \varphi, \varphi\rangle_D \end{aligned}$$

We will show that $(I + \Delta)$ is invertible and that its inverse satisfies the constraints of the spectral theorem. This begins with the following lemma:
Lemma (Gårding inequality)
There exists some C such that $\|\varphi\|_1^2 \leq C\|\varphi\|_D^2$.

Evidently, $\|\varphi\|_0^2 \leq \|\varphi\|_D^2$, so the following operator is bounded with respect to the Dirichlet norm:

$$T_\alpha : \varphi \mapsto \langle \alpha, \varphi \rangle$$

In the appropriate Hilbert space, we may use the Hilbert representation theorem to attain a bounded linear operator T such that:

$$\langle T\alpha, \varphi \rangle_D = \langle \alpha, \varphi \rangle$$

Hence $(I + \Delta)T\alpha = \alpha$. Now by the Gårding inequality:

$$\begin{aligned}
\|T\alpha\|_1^2 &\leq C\|T\alpha\|_D^2 \\
&= C\langle \alpha, T\alpha \rangle \\
&\leq C\|\alpha\|_0\|T\alpha\|_0 \\
&\leq \frac{1}{2}(\varepsilon C\|T\alpha\|_0^2 + \varepsilon^{-1}C\|\alpha\|_0^2) \\
&\leq \frac{1}{2}(\varepsilon C\|T\alpha\|_1^2 + \varepsilon^{-1}C\|\alpha\|_0^2) \\
\|T\alpha\|_1^2 &\leq C'\|\alpha\|_0^2
\end{aligned}$$

Hence T is a bounded linear operator from $\mathbf{H}_0$ to $\mathbf{H}_1$. By the Rellich lemma, it is then a compact operator from $\mathbf{H}_0$ to itself. It is self-adjoint, since $(I + \Delta)$ is self-adjoint. Therefore, by the spectral theorem, it is of the form

$$T\alpha = \sum_i \lambda_i \pi_i \alpha$$

where λ_i is an eigenvalue, and π_i is a projection onto the appropriate eigenspace. Furthermore, we also know that each eigenspace is finite-dimensional.
Since $(I + \Delta)$ is strictly positive definite, it is injective. But we also know that T is its right-inverse. It is therefore bijective, and $T = (I + \Delta)^{-1}$. Hence Δ has the same eigenspaces as T, with the eigenvalues $\rho_i = \frac{1-\lambda_i}{\lambda_i}$. Putting it all together:

Lemma (Spectral decomposition of Δ)

On a compact manifold, the operator Δ defines an orthonormal basis of eigenforms, and each eigenspace is finite-dimensional.

Our goal was to show that harmonic forms are smooth. For this, we use two lemmas:

Lemma (Sobolev)

In an n-dimensional space, $\mathbf{H}_{s+\lfloor \frac{n}{2} \rfloor+1} \subset C^s$.

Lemma (Regularity)

If $\Delta\psi = \varphi$ and $\varphi \in \mathbf{H}_s$ and $\psi \in \mathbf{H}_0$, then $\psi \in \mathbf{H}_{s+2}$.

From these, it follows that the eigenspace associated with any eigenvalue of Δ consists of smooth forms. As a special case, $\mathcal{H}^{p,q} \subset A^{p,q}$. This is what we wanted to show.

So we have that the solution to the minimum norm problem defines a harmonic form to represent every cohomology class. It is unique since

$$\begin{aligned}
\Delta\eta = 0 &\implies \langle \Delta\eta, \eta \rangle = \|\bar{\partial}\eta\|^2 + \|\bar{\partial}^*\eta\|^2 = 0 \\
&\implies \bar{\partial}\eta = 0 \land \bar{\partial}^*\eta = 0
\end{aligned}$$

and as we have already discussed, $\bar{\partial}^*\eta = 0$ implies that η is the solution to the minimum norm problem, of which there is only one. This also implies that every harmonic form defines a cohomology class. Clearly, this bijection is linear, from which we have the Hodge theorem.

Theorem (Hodge)

On a compact Hermitian manifold, $H^{p,q}_{\bar{\partial}} \simeq \mathcal{H}^{p,q}$, and it is finite-dimensional.

In general, we define the Hodge numbers of a complex manifold as

$$h^{p,q}(M) = \dim(H^{p,q}_{\bar{\partial}}(M))$$

This is the basic result of Hodge theory. They allows us to classify complex manifolds into many classes. The Hodge numbers are typically presented graphically in the so-called Hodge diamond:

$$\begin{array}{ccccc} & & h^{n,n} & & \\ & h^{n,n-1} & & h^{n-1,n} & \\ \vdots & & \vdots & & \vdots \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array}$$

For example, the Hodge diamond of the complex 2-torus ($T^2 = \mathbb{C}^2/\Lambda$ where Λ is some non-degenerate lattice acting on the space via translation) is

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array}$$

This data is far from enough to fully discern two manifolds. There exist compact manifolds with the same Hodge numbers that are not diffeomorphic.[2]

Here, we also note that the same proof works for the d-Laplacian and the ∂-Laplacian. Therefore we have the statements

Theorem (Hodge)

On a compact Hermitian manifold, $H^{p,q}_{\partial} \simeq \mathcal{H}^{p,q}_{\partial}$, and it is finite-dimensional.

On a compact Riemann manifold, $H^r_{\mathrm{DR}} \simeq \mathcal{H}^r_d$, and it is finite-dimensional.

2.2 Hodge decomposition

We have shown that Δ is of the form

$$\Delta\alpha = \sum_i \lambda_i \pi_i \alpha$$

where π_i are projections to finite-dimensional eigenspaces. We may define the so-called Green's operator by

$$G\alpha = \sum_{\lambda_i \neq 0} \lambda_i^{-1} \pi_i \alpha$$

Since $\mathcal{H}^{p,q}$ is finite-dimensional, it is a closed subspace of $A^{p,q}$. This allows us to define a projection operator π to that subspace. With the above definition, we have:

$$\Delta G\psi = (I - \pi)\psi$$

This gets us the decomposition

$$\begin{aligned} \psi &= \pi\psi + \Delta G\psi \\ &= \pi\psi + \bar{\partial}(\bar{\partial}^* G\psi) + \bar{\partial}^*(\bar{\partial} G\psi) \end{aligned}$$

In fact, this reflects something more fundamental:
Theorem (Hodge decomposition)
One has the following orthogonal decomposition:

$$A^{p,q} = \mathcal{H}^{p,q} \oplus \bar{\partial} A^{p,q-1} \oplus \bar{\partial}^* A^{p,q+1}$$

The projections to these components are given by π, $\bar{\partial}\bar{\partial}^* G$, and $\bar{\partial}^*\bar{\partial} G$.

The orthogonality is easy to see. The only non-trivial part at this point is showing that each decomposition is unique. Suppose that we have

$$0 = \eta + \bar{\partial}\psi + \bar{\partial}^*\varphi \text{ where } \eta \in \mathcal{H}^{p,q}$$

Taking the derivative of both sides, we find that $\bar{\partial}\bar{\partial}^*\varphi = 0$. By adjointness, $\mathrm{Im}(\bar{\partial}^*) = \mathrm{Ker}(\bar{\partial})^{\perp}$. Therefore, $\bar{\partial}^*\varphi = 0$. Likewise, $\bar{\partial}\psi = 0$, from which it follows that $\eta = 0$. So the decomposition holds.

3 The structure of the harmonic space

We have established that the harmonic space encodes the geometric information of the Dolbeault cohomology. Therefore, it is worth to examine its structure.

3.1 Serre duality

First, we have that the commutator $[\Delta, *]$ is 0, and therefore that $*$ induces an isomorphism $\mathcal{H}^{p,q} \simeq \mathcal{H}^{n-p,n-q}$. Immediately, this shows the following theorem:
Theorem (Serre duality)
The Hodge star operator induces an isomorphism:

$$H_{\bar{\partial}}^{p,q} \simeq H_{\bar{\partial}}^{n-p,n-q}$$

In terms of Hodge numbers:

$$h^{p,q} = h^{n-p,n-q}$$

For example, on a connected manifold, $H_{\bar{\partial}}^{n,n} \simeq \mathbb{C}$, consisting of constant multiples of the volume form. The isomorphism is given by

$$\psi \mapsto \int_M \psi$$

That this doesn't depend on the chosen representative may be seen using Stokes's theorem. In general, $d = \partial + \bar{\partial}$, but on $A^{n,n-1}$ specifically, $\partial = 0$, since $(n+1, n-1)$-forms don't exist. Since representatives can only differ by some $\eta \in \mathrm{Im}(\bar{\partial}) = \mathrm{Im}(d)$, Stokes's theorem implies they cancel out.
In general, one also has a pairing of the form

$$H_{\bar{\partial}}^{p,q} \otimes H_{\bar{\partial}}^{n-p,n-q} \longrightarrow \mathbb{C}$$

given by

$$(\psi, \eta) \mapsto \int_M \psi \wedge \eta$$

It is non-degenerate, since $(\psi, *\psi)$ maps to $\|\psi\|^2 \neq 0$.

3.2 Künneth theorem

Given the cohomologies of M and N, it is natural to ask about $M \times N$. The Hodge theorem gives us a particularly simple way to show that
Theorem (Künneth)

$$H_{\bar{\partial}}^{p,q}(M \times N) \simeq \bigoplus_{\substack{k+l=p \\ m+n=q}} \left(H_{\bar{\partial}}^{k,m}(M) \otimes H_{\bar{\partial}}^{l,n}(N) \right)$$

In terms of Hodge numbers:

$$h^{p,q}(M \times N) = \sum_{\substack{k+l=p \\ m+n=q}} \left(h^{k,n}(M) h^{l,m}(N) \right)$$

To see this, we may just have to show that harmonic spaces of $M \times N$ decompose into products of harmonic spaces on M and N.

Let us call a form on $M \times N$ decomposable if it is the product of forms on M and N. One can easily see in local coordinates that:

$$\bar{\partial}_{M\times N}(\alpha \wedge \beta) = \bar{\partial}_M \alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \bar{\partial}_N \beta$$

Furthermore, for the Hodge star operator:

$$\begin{aligned}
(\psi \wedge \eta) \wedge *_{M\times N}(\alpha \wedge \beta) &= \langle \psi \wedge \eta, \alpha \wedge \beta \rangle_{M\times N} \mathrm{vol}_{M\times N} \\
&= \langle \psi, \alpha \rangle_M \langle \eta, \beta \rangle_N \mathrm{vol}_M \wedge \mathrm{vol}_N \\
&= (\psi \wedge *_M \alpha) \wedge (\eta \wedge *_N \beta) \\
&= (-1)^{(m-k)l} (\psi \wedge \eta) \wedge (*_M \alpha \wedge *_N \beta) \\
*_{M\times N}(\alpha \wedge \beta) &= (-1)^{(m-k)l} (*_M \alpha) \wedge (*_N \beta)
\end{aligned}$$

where m is the number of dimensions in M, k is the order of α, and l is the order of β.
Working out the dimensions, we attain that

$$\bar{\partial}^*_{M\times N}(\alpha \wedge \beta) = \bar{\partial}^*_M \alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \bar{\partial}^*_N \beta$$

In the Laplace operator, the signs cancel out nicely and we attain

$$\Delta_{M\times N}(\alpha \wedge \beta) = \Delta_M \alpha \wedge \beta + \alpha \wedge \Delta_N \beta$$

As it turns out, the set of decomposable forms is total. This can be seen easily: One may construct a decomposable bump function on an arbitrarily small neighbourhood of any point. The totality of this subset lets us extend our results of decomposable forms to the general case.
If $\{\psi_i\}_i$ is a complete set of eigenforms of $A^{p,q}(M)$ and $\{\eta_j\}_j$ is a complete set of eigenforms of $A^{r,s}(N)$, then $\{\psi_i \wedge \eta_j\}_{i,j}$ will be a complete set of eigenforms in $A^{p+r,q+s}(M \times N)$. We may calculate the associated eigenvalues:

$$\begin{aligned}
\Delta_{M\times N}(\psi_i \wedge \eta_j) &= \Delta_M \psi_i \wedge \eta_j + \psi_i \wedge \Delta_N \eta_j \\
&= (\lambda_i + \mu_j)(\psi_i \wedge \eta_j)
\end{aligned}$$

So $\lambda_i + \mu_j$ is the eigenvalue associated with $\psi_i \wedge \eta_j$.
Since Δ is positive-semidefinite, we have

$$\lambda_i + \mu_j = 0 \Longleftrightarrow \lambda_i = 0 \wedge \mu_j = 0$$

which gives us the Künneth theorem, since the kernel space decomposes as we would expect.

4 Kähler manifolds

We call a manifold Euclidean if every point has a neighbourhood with a system of local coordinates such that:

$$ds^2 = \sum_j dz_j \otimes d\bar{z}_j$$

This is quite restrictive, and it gives rise to a number of symmetries, which will be discussed later. We wish to define a more relaxed condition that still permits these symmetries. A metric satisfying this more relaxed condition will be called a **Kähler meetric**. A manifold with a Kähler metric (or by some authors, a manifold that admits a Kähler metric) is called a Kähler manifold.

There are three main and equivalent ways to define a Kähler metric:
Definition (Kähler metrics)
Let ds^2 be a metric on the tangent bundle. Then, the following are equivalent:

- The associated $(1,1)$ form is d-closed.
- The Chern connection is torsion-free.
- The metric osculates to order 2 to a Euclidean metric everywhere. That is, around any point p, for some holomorphic coordinate system,

$$ds^2 = \sum_{j,k} (\delta_{jk} + f_{jk}) dz_j \otimes d\bar{z}_k$$

  where f_{jk} vanishes to order 2 at p. We call these "normal coordinates" around p.

The metric is Kähler if any of these are satisfied.

Seeing the first two from the third is easy. They depend only on the first derivatives of the metric, which vanish at every point.

To see why the first and the second are fulfilled simultaneously, notice first that $d\omega = 0 \Leftrightarrow \partial\omega = 0$, since $d = \partial + \bar{\partial}$ and $\omega = \bar{\omega}$. The calculation of $\partial\omega$ for some holomorphic coordinate frame is:

$$\begin{aligned}\partial\omega &= \partial \frac{i}{2} \sum_{i,j} h_{i\bar{j}} dz_i \wedge d\bar{z}_j \\ &= \frac{i}{2} \sum_{i,j,k} (\partial_k h_{i\bar{j}}) dz_k \wedge dz_i \wedge d\bar{z}_j\end{aligned}$$

Meanwhile, the calculation of the torsion is as follow. Note that since we are working in a coordinate frame, $[e_i, e_j] = 0$.

$$\begin{aligned}T(e_i, e_j) &= \nabla_{e_i} e_j - \nabla_{e_j} e_i - [e_i, e_j] \\ &= \sum_{k,l} h^{k\bar{l}} (\partial_i h_{j\bar{l}} - \partial_j h_{i\bar{l}}) e_k\end{aligned}$$

These two encode the exact same data.

On the other hand, it can be shown that when $d\omega = 0$, any holomorphic coordinate system around p can be transformed into an appropriate system to show that the metric osculates to a Euclidean metric.

Defining the Betti numbers $b_m(M) = \dim(H^m_{\text{DR}}(M))$, we have one immediate consequence of a manifold being Kähler:
Theorem (Positivity of even Betti numbers on Kähler manifolds)
On a Kähler manifold, $b_{2k} \geq 1$ for all $k \leq n$.

The nontriviality of $H^m_{\text{DR}}(M)$ is exhibited by the exterior powers of ω. Since ω is d-closed and not exact, so is $\omega^{\wedge k}$. This construction remains valid as long as $k \leq n$.

4.1 Kähler identities

On any complex Hermitian manifold, we may define a number of operators. d, ∂, and $\bar{\partial}$ we already know. We also have

$$d^c = -\frac{i}{2}(\partial - \bar{\partial})$$

This operator simplifies a number of identities in the theory, none of which will be mentioned here. It is being included for the sake of completeness. Each operator also have adjoints d^*, ∂^*, $\bar{\partial}^*$, and d^{c*}.
We also have the Lefschetz operator

$$L(\alpha) = \alpha \wedge \omega$$

and its adjoint Λ.
We also have the Laplacians

$$\Delta_d = dd^* + d^*d$$
$$\Delta_\partial = \partial\partial^* + \partial^*\partial$$
$$\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$$

and the projection operators

$$\Pi^{p,q} : A^* \to A^{p,q}$$

In general, these operators don't have that many relationships among one another. However, on a Kähler manifold, we find that their commutators are remarkably well-behaved.
Theorem
On a Kähler manifold:

$$[\bar{\partial}, L] = 0 \qquad [\partial, L] = 0 \qquad [\bar{\partial}^*, \Lambda] = 0 \qquad [\partial^*, \Lambda] = 0$$
$$[\bar{\partial}, \Lambda] = -i\bar{\partial}^* \qquad [\partial, \Lambda] = i\partial^* \qquad [\bar{\partial}^*, L] = i\partial \qquad [\partial^*, L] = -i\bar{\partial}$$
$$[d, \Lambda] = -2d^{c*} \qquad [d^{c*}, \Lambda] = 0 \qquad [d^*, L] = -2d^c \qquad [d, L] = 0$$

Furthemore,

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$$

and each Laplacian commutes with $*$, ∂, $\bar{\partial}$, $\bar{\partial}$, $\bar{\partial}^*$, L, Λ and $\Pi^{p,q}$.

The proof for the first 12 are similar. One first proves it on $\mathbb{C}^n$, where it is a rote calculation. Then, since we are on a Kähler manifold, we may use normal coordinates around any point. In normal coordinates, the first derivatives behave exactly as they would in $\mathbb{C}^n$, so we observe that the proof works in every point.
The proofs for the identities involving the Laplacians are corollaries of these.

4.2 Hodge decomposition

The most useful consequence of these identities is the Hodge decomposition:
Theorem (Hodge decomposition)
On a compact Kähler manifold,

$$H^r_{\text{DR}}(M) \simeq \bigoplus_{p+q=r} H^{p,q}_{\bar\partial}(M)$$
$$H^{p,q}_{\bar\partial}(M) = \overline{H^{q,p}_{\bar\partial}(M)}$$

The first comes from the fact that $\Delta_d = 2\Delta_{\bar\partial}$ commutes with $\Pi^{p,q}$. The projections of harmonic functions are exactly the harmonic ones in the projected space, therefore

$$\begin{aligned}\mathcal{H}^r &= \bigoplus_{p+q=r} \Pi^{p,q}\mathcal{H}^r \\ &= \bigoplus_{p+q=r} \mathcal{H}^{p,q}\end{aligned}$$

From the Hodge theorem, we have the first part of the statement. The second part is given by the fact that Δ_d is a real operator. Therefore $\Delta_d \eta = 0 \Leftrightarrow \Delta_d \overline{\eta} = 0$.

We may definine the Betti numbers $b^r(M) = \dim(H^r_{\text{DR}}(M))$. In terms of these, we have
Corollary (Betti and Hodge numbers on Kähler manifolds)
On a Kähler manifold,

$$b^r = \bigoplus_{p+q=r} h^{p,q}$$
$$h^{p,q} = h^{q,p}$$

4.3 Lefschetz decomposition

The second result we have comes from representation theory. The Lie algebra $\mathfrak{sl}_2$ is the one associated with the Lie group SL_2. It is generated by the elements

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

These satisfy the Lie algebra relations

$$[X, Y] = H \qquad [H, X] = 2X \qquad [H, Y] = -2Y$$

A collection of three elements in some Lie algebra $\mathfrak{g}$ satisfying this property in a Lie algebra is called an $\mathfrak{sl}_2$-triple. They generate Lie algebra homomorphisms $\mathfrak{sl}_2 \to \mathfrak{g}$.

We call a Lie algebra homomorphism $\rho : \mathfrak{sl}_2 \to \mathfrak{gl}(V)$ a representation of $\mathfrak{sl}_2$ in V. V, together with such a representation, is called an $\mathfrak{sl}_2$-module. We call a subspace of V fixed under the actions of $\mathfrak{sl}_2$ a submodule. It is known that every finite-dimensional $\mathfrak{sl}_2$-module is the direct sum of irreducible submodules.

It is also well-known that there is exactly one $\mathfrak{sl}_2$-module of any finite dimension. One realisation of an $n+1$-dimensional $\mathfrak{sl}_2$ module is that of the space of homogeneous polynomials in 2 variables. Here, the actions

$$\rho(X)(p) = x\frac{\partial p}{\partial y} \qquad \rho(H)(p) = x\frac{\partial p}{\partial x} - y\frac{\partial p}{\partial y} \qquad \rho(Y)(p) = y\frac{\partial p}{\partial x}$$

define the structure of this module. One can easily see then that the space V decomposes into the so-called weight spaces; the eigenspaces of $\rho(H)$. Each eigenvalue is integer. For an $n+1$-dimensional V, this looks like

$$V = V_{-n} \oplus V_{-n+2} \oplus \cdots \oplus V_{n-2} \oplus V_n$$

Furthermore, $\rho(X)$ and $\rho(Y)$ act on these subspaces by

$$\rho(X) : V_m \to V_{m+2}$$
$$\rho(Y) : V_m \to V_{m-2}$$

Extending this result to not necessarily irreducible V, we attain the following result:
Theorem (Structure of finite-dimensional $\mathfrak{sl}_2$-modules)
Any $\mathfrak{sl}_2$-module (V, ρ) decomposes into into eigenspaces of $\rho(H)$ of integer eigenvalue. The appropriate restrictions of the operators send

$$\rho(X) : V_m \to V_{m+2}$$
$$\rho(Y) : V_m \to V_{m-2}$$

Furthermore,

$$\forall k \leq m : \rho(X)^k : V_{-m} \to V_{-m+2k}$$
$$\forall k \leq m : \rho(Y)^k : V_m \to V_{m-2k}$$

are injections. When $k = m$, they are isomorphisms.

There is also a different way of looking at this. We call an element $v \in V$ primitive when it is not $\mathrm{Im}(\rho(X))$; or equivalently, when it is not in $\mathrm{Ker}(\rho(Y))$. We call the space of primitive elements in V_m by P_m. Then we have the decomposition $V_m = \rho(X)V_{m-2} \oplus P_m$. Then we may decompose V_{m-2} further, attaining an expression with V_{m-4}, and so on. Since V is finite-dimensional, this eventually ends, and we attain

$$V_m = \bigoplus_{k \geq 0} \rho(X)^k P_{m-2k}$$

This is what we will later call the Lefschetz decomposition.

Back to harmonic forms, it so happens that $\mathcal{H}_d = \mathrm{Ker}(\Delta_d)$ is an $\mathfrak{sl}_2$ module. One may easily verify that for a p-form η, $[\Lambda, L]\eta = (n-p)\eta$. Letting h be the operator that multiplies each component of degree p by $n-p$, we find that

$$[\Lambda, L] = h \qquad [h, \Lambda] = 2\Lambda \qquad [h, L] = -2L$$

The eigenspaces of h are, of course, the spaces of harmonic forms of pure degree, $\mathcal{H}_d^p$. We also know from the Hodge theorem that $\mathcal{H}_d^p \simeq H_{\mathrm{DR}}^p$. This allows us to create a decomposition of the de Rham cohomology.

Applying the general theory of $\mathfrak{sl}_2$-representations to $\mathcal{H}_d$, we get the
Theorem (Hard Lefschetz theorem)

The map

$$L^k : H^{n-k}_{\mathrm{DR}} \to H^{n+k}_{\mathrm{DR}}$$

is an isomorphism.

Furthermore, letting

$$\begin{aligned} P^{n-k} &= H^{n-k}_{\mathrm{DR}} \cap \mathrm{Ker}(\Lambda) \\ &= H^{n-k}_{\mathrm{DR}} \cap \mathrm{Ker}(L^{k+1}) \end{aligned}$$

we have the Lefschetz decomposition:

$$H^m_{\mathrm{DR}} = \bigoplus_{k \geq 0} L^k P^{m-2k}$$

The Lefschetz decomposition is compatible with the Hodge decomposition, since L, Λ and h each commute with $\Delta_{\bar{\partial}}$. Setting

$$P^{p,q} = \mathrm{Ker}(\Lambda) \cap H^{p,q}_{\bar{\partial}}$$

we have

$$P^r = \bigoplus_{p+q=r} P^{p,q}$$

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