

# Hodge theory

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## Complex geometry

A **complex manifold** is a manifold with an atlas going into  $\mathbb{C}^k$ , where the associated transition maps are holomorphic, in the sense of multi-dimensional complex analysis.

The subspaces spanned locally in the tangent space by the **Wirtinger derivatives**  $\{\partial/\partial z_i\}$  and  $\{\partial/\partial \bar{z}_i\}$  are preserved by transition maps. They therefore define a decomposition of the tangent map into the holomorphic and antiholomorphic parts

$$TM = T'M \oplus T''M$$

The cotangent bundle likewise decomposes into the corresponding dual spaces. As such so do the spaces of differential forms. This gives us the bigrading:

$$A^{p,q}(M) = (T'^*M)^{\wedge p} \wedge (T''^*M)^{\wedge q}$$

## Complex geometry

$$A^{p,q}(M) = (T^*M)^{\wedge p} \wedge (T^*M)^{\wedge q}$$

The restriction of ordinary exterior derivative goes

$$d : A^{p,q} \longrightarrow A^{p+1,q} \oplus A^{p,q+1}$$

This decomposes into the **Dolbeault operators**

$$d = \partial + \bar{\partial}$$

$$\partial : A^{p,q} \rightarrow A^{p+1,q}$$

$$\bar{\partial} : A^{p,q} \rightarrow A^{p,q+1}$$

## Complex geometry

These three operators define cochain complexes. Defining

$$A^k = \bigoplus_{p+q=k} A^{p,q}$$

we have  $d$  going

$$A^0 \xrightarrow{d} A^1 \xrightarrow{d} \dots$$

and the other two operators going

$$A^{0,q} \xrightarrow{\partial} A^{1,q} \xrightarrow{\partial} \dots$$

$$A^{p,0} \xrightarrow{\bar{\partial}} A^{p,0} \xrightarrow{\bar{\partial}} \dots$$

with  $d^2 = \partial^2 = \bar{\partial}^2 = 0$ .

# Complex geometry

$$\begin{aligned}A^0 &\xrightarrow{d} A^1 \xrightarrow{d} \dots \\A^{0,q} &\xrightarrow{\partial} A^{1,q} \xrightarrow{\partial} \dots \\A^{p,0} &\xrightarrow{\bar{\partial}} A^{p,1} \xrightarrow{\bar{\partial}} \dots \\d^2 &= \partial^2 = \bar{\partial}^2 = 0\end{aligned}$$

Given the cochain complexes above, we define the **de Rham cohomology** and the **Dolbeault cohomology** as

$$\begin{aligned}H_{\text{DR}}^k &= \frac{\text{Ker}(d : A^k \rightarrow A^{k+1})}{\text{Im}(d : A^{k-1} \rightarrow A^k)} \\H_{\bar{\partial}}^{p,q}(M) &= \frac{\text{Ker}(\bar{\partial} : A^{p,q} \rightarrow A^{p,q+1})}{\text{Im}(\bar{\partial} : A^{p,q-1} \rightarrow A^{p,q})}\end{aligned}$$

## Hermitian geometry

A hermitian manifold is a complex manifold together with a hermitian metric on its holomorphic tangent bundle. A hermitian metric is a smoothly-varying positive-definite hermitian form. So a metric  $h$  in a point  $p$  takes

$$h_p : T'_p M \times T'_p M \rightarrow \mathbb{C}$$

such that  $h_p$  is sesquilinear and  $h_p(v, w) = \overline{h_p(w, v)}$ .

A hermitian metric defines an ordinary Riemann metric on the underlying real manifold. As such, each Hermitian metric also has an associated volume form. If one has, in local coordinates, a unitary coframe  $\{\varphi_i\}$ , then the volume form is

$$d\mu = \left(\frac{i}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_n$$

## Hermitian geometry

Given a hermitian metric on the tangent bundle, we may define metrics on the bundles of differential forms. In local coordinates, if  $\{\varphi_i\}$  is the dual of a unitary frame, we define a metric on  $A^{p,q}$  by setting

$$\begin{aligned}\{\varphi_I \wedge \bar{\varphi}_J\}_{|I|=p, |J|=q} &\text{ is unitary} \\ \|\varphi_I \wedge \bar{\varphi}_J\| &= 2^{p+q}\end{aligned}$$

We thus get an operator

$$h : A^{p,q} \times A^{p,q} \rightarrow C^\infty(M)$$

where  $C^\infty(M)$  is the space of smooth functions  $M \rightarrow \mathbb{C}$ .

## Adjoint differential operators

Taking the integral of the induced metric lets us turn the space  $A^{p,q}$  into a pre-Hilbert space with the inner product

$$\langle \psi, \eta \rangle = \int_M h_p(\psi, \eta) d\mu(p)$$

Here, the differential operators will have adjoints. To define them concretely, we may introduce the **Hodge star operator**  $*$ , going  $A^{p,q} \rightarrow A^{n-p, n-q}$ . It is defined by the relation

$$\int (\eta \wedge \psi) d\mu = \int \mu \wedge * \psi$$

The adjoints are then

$$d^* = -*d*$$

$$\partial^* = -*\partial*$$

$$\bar{\partial}^* = -*\bar{\partial}*$$

# Adjoint differential operators

The adjoint differential operators are of types

$$d^* : A^{p,q} \rightarrow A^{p-1,q} \oplus A^{p,q-1}$$

$$\partial^* : A^{p,q} \rightarrow A^{p-1,q}$$

$$\bar{\partial}^* : A^{p,q} \rightarrow A^{p,q-1}$$

and they fulfil the relations

$$\langle \partial\psi, \eta \rangle = \langle \psi, \partial^*\eta \rangle$$

$$\langle \bar{\partial}\psi, \eta \rangle = \langle \psi, \bar{\partial}^*\eta \rangle$$

$$\langle d\psi, \eta \rangle = \langle \psi, d^*\eta \rangle$$

Just like the differential operators induce cochain complexes, their adjoints induce chain complexes in the other directions.

# Laplacians

Using the previously defined operators, we may define 3 Laplace operators. The **de Rham Laplace operator** is

$$\Delta_d = d^*d + dd^* = (d + d^*)^2$$

The  $\partial$ -**Laplace operator** and  $\bar{\partial}$ -**Laplace operator** are likewise

$$\Delta_\partial = \partial^*\partial + \partial\partial^* = (\partial + \partial^*)^2$$

$$\Delta_{\bar{\partial}} = \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^* = (\bar{\partial} + \bar{\partial}^*)^2$$

We call the kernels of these operators the **harmonic spaces**. These are also graded. We define

$$\begin{aligned}\mathcal{H}_d^k &= \text{Ker}(\Delta_d) \cap A^k \\ \mathcal{H}_\partial^{p,q} &= \text{Ker}(\Delta_\partial) \cap A^{p,q} \\ \mathcal{H}_{\bar{\partial}}^{p,q} &= \text{Ker}(\Delta_{\bar{\partial}}) \cap A^{p,q}\end{aligned}$$

## Cohomologies as affine subspaces

As we know, the de Rham and Dolbeault cohomologies, respectively, are derived from the cochains induced by  $d$  and  $\bar{\partial}$ .

$$H_{\text{DR}}^k = \frac{\text{Ker}(d : A^k \rightarrow A^{k+1})}{\text{Im}(d : A^{k-1} \rightarrow A^k)}$$
$$H_{\bar{\partial}}^{p,q}(M) = \frac{\text{Ker}(\bar{\partial} : A^{p,q} \rightarrow A^{p,q+1})}{\text{Im}(\bar{\partial} : A^{p,q-1} \rightarrow A^{p,q})}$$

Another way to look at this is that elements of these cohomologies may be represented, respectively, as affine subspaces of forms

$$\psi + dA^{k-1} \subset A^k \quad \text{where } \psi \in \text{Ker}(d)$$
$$\psi + \bar{\partial}A^{p,1-q} \subset A^{p,q} \quad \text{where } \psi \in \text{Ker}(\bar{\partial})$$

It is natural to wonder if there is some natural way to represent each subspace.

## Harmonic forms as representatives of cohomologies

If one wishes to select a representative from an affine subspace of a Hilbert space, then one natural choice would be choosing the minimum norm element. The obstacle here is that the spaces of differential forms together with  $\langle \cdot, \cdot \rangle$  are only pre-Hilbert. However, in the real case, it can be seen that

### Lemma

*On a compact Riemann manifold, if  $\psi \in \text{Ker}(d)$ , then  $\eta \in \psi + dA^{k-1}$  is of minimum norm if and only if  $\eta \in \text{Ker}(d^*)$ . Furthermore, this is equivalent with stating that  $\eta \in \mathcal{H}_d^k$ .*

Likewise in the complex case,

### Lemma

*On a compact hermitian manifold, if  $\psi \in \text{Ker}(\bar{\partial})$ , then  $\eta \in \psi + \bar{\partial}A^{p,q-1}$  is of minimum norm if and only if  $\eta \in \text{Ker}(\bar{\partial}^*)$ . Furthermore, this is equivalent with stating that  $\eta \in \mathcal{H}_{\bar{\partial}}^{p,q}$ .*

# Hodge theorem

The fact that minimum norm representatives always have to be harmonic, together with the (non-trivial) observation that we can always choose a harmonic representative nets us the Hodge theorem.

## Theorem (Hodge)

*On a compact Riemann manifold,*

$$\mathcal{H}_d^k \simeq H_{\text{DR}}^k$$

*On a compact hermitian manifold,*

$$\mathcal{H}_{\bar{\partial}}^{p,q} \simeq H_{\bar{\partial}}^{p,q}$$

It should also be noted that the right side does not depend on the metrics chosen, only the real/complex structure of the manifold.

# Invariants

$$\mathcal{H}_d^k \simeq H_{\text{DR}}^k$$
$$\mathcal{H}_{\bar{\partial}}^{p,q} \simeq H_{\bar{\partial}}^{p,q}$$

On a compact manifold, these harmonic spaces are finite-dimensional. The dimensions of the de Rham cohomology groups are the **Betti numbers**, denoted  $b_k$ , and the dimensions of the Dolbault cohomology groups the **Hodge numbers**, denoted  $h^{p,q}$ . The Hodge numbers are typically represented graphically via the Hodge diamond:

$$\begin{array}{ccccc} & & h^{n,n} & & \\ & h^{n,n-1} & & h^{n-1,n} & \\ & \vdots & & \vdots & \vdots \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array}$$

## Invariants

For example, the Hodge diamond of the complex 2-torus (which has 4 real dimensions) is

$$\begin{array}{ccccc} & & & & 1 \\ & & & & 2 & & 2 \\ & & & 1 & & 4 & & 1 \\ & & & & & 2 & & 2 \\ & & & & & & & & 1 \end{array}$$

This allows us to distinguish complex manifolds from one another.

# Consequences

The isomorphism of the harmonic spaces and the cohomology groups can be used to prove a number of theorems. Here, we list two.

## Theorem (Serre duality)

*On a compact complex manifold,*

$$h^{p,q} = h^{n-p,n-q}$$

## Theorem (Künneth formula)

*Given two compact complex manifolds  $M$  and  $N$ ,*

$$h^{p,q}(M \times N) = \sum_{\substack{k+l=p \\ m+n=q}} (h^{k,n}(M)h^{l,m}(N))$$

## Consequences: Kähler manifolds

We call a hermitian manifold **Kähler** if, loosely, its metric can be locally approximated with the canonical metric of  $\mathbb{C}^n$ . In this case, the analytical properties simplify greatly. Since we can use the Laplacians to study the geometry, this has geometric consequences as well. Most importantly, we have the

### Theorem (Hodge decomposition)

*On a compact Kähler manifold, the de Rham cohomology decomposes into the appropriate levels of the Dolbeault cohomology:*

$$H_{\text{DR}}^k \simeq \bigoplus_{p+q=k} H_{\bar{\partial}}^{p,q}$$

*Furthermore, conjugation induces an automorphism of the Dolbeault cohomology:*

$$H_{\bar{\partial}}^{p,q} = \overline{H_{\bar{\partial}}^{q,p}}$$

Thank you for listening.

During the project, the free plan of ChatGPT was used for research. No other AI tools were used, and for no other purpose.