Alternative definition for the ribbon graph polynomial

The Tutte polynomial is a 2-variable graph polynomial defined for every undirected graph. There are several known definitions of the polynomial. In 2006, Bernardi [1] gave a characterization using tours of spanning trees.

Bollobás and Riordan introduced a 4-variable polynomial for embedded graphs. A 2-variable specialization of this polynomial is called the Tutte polynomial of cellularly embedded graphs or the ribbon graph polynomial. From now on we will refer to the Tutte polynomial of abstract graphs as Tutte polynomial, and to the Tutte polynomial of cellularly embedded graphs as ribbon graph polynomial.

Similarly to the Tutte polynomial of abstract graphs, the ribbon graph polynomial has several equivalent definitions: through deletion-contraction recursion, by a sum on all subsets of edges using rank and nullity functions, or using activities of quasi-trees. The latter definition is somewhat artificial, because it requires fixing an ordering of the edges (that the polynomial does not depend on). In this paper, we show that one can adapt Bernardi's approach to the Tutte polynomial of abstract graphs, and define activities in a way, that only require fixing a vertex and an incident edge of the embedded graph.

1. Introduction

Let G be a connected, undirected graph. We fix a cyclic order of the incident edges for all of the vertices in G. This is called a ribbon structure. It is known that for any G graph and ribbon structure there exists an orientable surface Σ into which it can be embedded. In this paper we only consider ribbon graphs that can be embedded to oriented surfaces thus they do not contain twisted ribbons (homeomorphic to a Möbius band). We call it a cellural embedding to Σ if the regions determined by the edges and vertices are homeomorphic to discs. Let us denote the embedded ribbon graph by $\mathbb{G}$. We define the dual of ribbons graph embedded in a surface by taking the regions as vertices and connecting two if they have a common edge. It follows immediately that the dual can be cellularly embedded to the same surface.

One of the definitions (activities expansion) of the Tutte polynomial uses activities on the edges. First we label the edges from 1 to $|E|$. For a spanning tree, we say that an edge e contained in a tree is internally active if it has the largest label among the edges in its fundamental cut. (Deleting e from the tree creates two component. The cut include the edges running between the components.) An edge is externally active if it has the largest label among the edges in its fundamental cycle. We denote the number of internally and externally active edges for a spanning tree T with $IA(T)$ and $EA(T)$ correspondingly. Then the Tutte polynomial of a G graph is:

$$T(G;x,y) = \sum_{T \text{ spanning tree}} x^{IA(T)} y^{EA(T)}$$

It can be shown that the polynomial obtained is independent from the initial numbering of edges.

Bernardi defined the embedding-activities with a tour of the spanning tree. First we take an arbitrary embedding of the graph to the plane and the corresponding ribbon graph. Here, we put discs in place of the vertices of the ribbon graph, and ribbons in place of the edges. Then fix a starting point on the boundary and a direction and take a walk around the spanning tree (only crossing the external edges). Therefore we obtain a sequence of the edges, each appearing two times.

Take the sequence obtained with the tour of the tree. We call two edges interlacing if they appear alternating in the sequence: $...e_1...e_2...e_1...e_2...$. Note that the edges in the same cut as an internal edge e are exactly the edges which are interlacing with e. For an external edge f the interlacing edges are exactly the edges of the fundamental cycle. Bernardi proved that if for every tree we label the edges in the order of their first appearance in the tour (using the same starting point for each tree) and define the embedding-activities with this numbering, we will obtain the same polynomial. Note that in Bernardi's definition for each tree, we define the embedding-activities using the ordering corresponding to that specific tree. [1]

We call a subset of edges a quasi-tree if the corresponding ribbon graph has exactly one boundary component. It immediately follows, that a quasi-tree is always connected. A spanning quasi-tree is a quasi-tree which contains all vertices.

---

[1] Bernardi originally defined an edge embedding-active if has the smallest number among the interlacing edges. Notice, that if we take the tour around the tree in the opposite direction, then the edge which appeared first among the interlacing edges in the original walk will appear last. Thus we can define the active edges to be the ones with the largest labels.

2. Definitions of the ribbon graph polynomial

Moffatt [4] defined the activities in an embedded graph using quasi-trees. For a non-planar graph a quasi-tree can contain circles, thus to define activities we use tours of quasi-trees. We fix a numbering of the edges. A spanning quasi-tree has exactly one boundary component thus we can take a tour and obtain a sequence of edges. An edge is externally/internally active if it is not contained/contained in the spanning quasi-tree $\mathbb{T}$ and has the largest label among the interlacing edges.

We will denote the ribbon graph polynomial by $T(\mathbb{G}; x, y)$. Moffatt defined the ribbon graph polynomial with the dicromatic polynomial for embedded graphs. The dicromathic polynomial can be expressed as a sum over all spanning quasi-trees with the following expression:

$$Z(\mathbb{G}; u, v) = v \sum_{\mathbb{T} \text{ spanning quasi-tree}} u^{e(\mathbb{T})} (1 + u^{-1} v)^{ILO(\mathbb{T})} (1 + uv)^{ELO(\mathbb{T})}$$

Here $e(\mathbb{T})$ is the number of edges in quasi-tree $\mathbb{T}$, $ILO(\mathbb{T})$ and $ELO(\mathbb{T})$ are the internally active and externally active edges of $\mathbb{T}$.

The ribbon graph polynomial can be expressed using the dichromatic polynomial $Z(\mathbb{G}; u, v)$ with the following expression:

$$T(\mathbb{G}, x + 1, y + 1) = \sqrt{x}^{e(\mathbb{G}) - b(\mathbb{G})} \cdot \sqrt{y}^{-v(\mathbb{G})} \cdot Z(\mathbb{G}; \sqrt{\frac{y}{x}}, \sqrt{xy})$$

Where $b(\mathbb{G})$ is the number of faces in the cellularly embedded ribbon graph $\mathbb{G}$, $e(\mathbb{G})$ and $v(\mathbb{G})$ are the number of edges and vertices of $\mathbb{G}$.

Now we can define the ribbon graph polynomial as a sum over all spanning quasi-trees of $\mathbb{G}$:

$$T(\mathbb{G}, x, y) = \sqrt{x - 1}^{e(\mathbb{G}) - b(\mathbb{G}) + 1} \cdot \sqrt{y - 1}^{1 - v(\mathbb{G})} \sum_{\mathbb{T}} \sqrt{\frac{y - 1}{x - 1}}^{e(\mathbb{T})} x^{ILO(\mathbb{T})} y^{ELO(\mathbb{T})}$$

Our aim is to show that the previous formula stays true if we define the activities with the order of the appearances of the edges in the tour $<_{\mathbb{T}}$. Here we only fix a starting vertex and edge, instead of fixing the order of the edges in advance, and summing on the trees using that order. For this we use another equivalent definition of the ribbon graph polynomial from Moffatt's article. We denote the dual graph $\mathbb{G}$ by $\mathbb{G}^*$ and the dual of an edge e by e^*.

$$T(\mathbb{G}, x, y) = f(e) T(\mathbb{G} \backslash e) + g(e) T(\mathbb{G}/e)$$

$$f(e) = \begin{cases} x - 1 & \text{if } e^* \text{ is a loop in } \mathbb{G}^* \\ 1 & \text{if } e^* \text{ is not a loop in } \mathbb{G}^* \end{cases} \tag{1}$$

$$g(e) = \begin{cases} y - 1 & \text{if } e \text{ is a loop in } \mathbb{G} \\ 1 & \text{if } e \text{ is not a loop in } \mathbb{G} \end{cases} \tag{2}$$

For the proof we examine if the recursion formula works for the activities defined by the reaching order in the tour for the different types of edges.

Since contraction and deletion operations are dual to each other, it is easy to show from the previous recursion formula that $T(\mathbb{G}; x, y) = T(\mathbb{G}^*; y, x)$.

In the following we would like to assume, that we can choose the starting point of the tour such that the edge e in the recursion is not active in any spanning quasi-trees. Obviously this is not true if for a quasi-tree there are no interlacing edges with e, because then e is active by definition. We will show that our assumtion is true for every e for which there exists a spanning quasi-tree containing it and a spanning quasi-tree not containing it.

Lemma 2.1 *An edge is contained in every/none of the spanning quasi-trees if and only if for arbitrary spanning quasi-tree there is no interlacing edge with it.*

Observe that if we have a spanning quasi-tree $\mathbb{T}$, an internal edge f_1 and an external edge f_2 interlacing with f_1, then $\mathbb{T} - f_1 + f_2$ is a spanning quasi-tree. Let us consider the sequence determined by the tour of $\mathbb{T}$ (let us start the tour along one side of f_1). The sequence is $f_1 \to a_1 \to f_2 \to a_2 \to f_1 \to a_3 \to f_2 \to a_4 \to$, where a_i is a series of edges. In this case the tour of $\mathbb{T} - f_1 + f_2$ will be $f_1 \to a_3 \to f_2 \to a_2 \to f_1 \to a_1 \to f_2 \to a_4$ thus the obtained graph $\mathbb{T} - f_1 + f_2$ has one boundary component. The same way if there is two external/internal interlacing edges then with including/deleting both we obtain a spanning quasi-tree. The new sequence of edges along the tour will be $f_1 \to a_3 \to f_2 \to a_2 \to f_1 \to a_1 \to f_2 \to a_4$ in both cases. It can be shown that every spanning quasi-tree can be obtained from each other with these steps.

If there exist a spanning quasi-tree in which an internal/external edge e is interlacing with an other edge, then with one of the steps described above we can construct a spanning quasi-tree where e is external/internal.

Take an edge e and a spanning quasi-tree $\mathbb{T}$ such that e has no interlacing edges in the tour of $\mathbb{T}$. Notice that in this case both of the instances of e is in the same a_i. Thus after any of the steps described above e still do not have interlacing edges. This way we showed, that if an edge has no interlacing edges in a spanning quasi-tree, then it has no interlacing edges in any spanning quasi-tree and thus it is contained in every/none of them. We proved the lemma.

Suppose that for an edge e there exist a spanning quasi-tree $\mathbb{T}$ such that in the tour of $\mathbb{T}$ there is an interlacing edge f. Then if we start the tour from one of the endpoints of e (along one side of e), then every interlacing edge appears later than e in the sequence, so e is not active in all of the spanning quasi-tree. Let us fix the starting vertex this way, so from now on we can suppose in our proof that either e has no interlacing edges or e is not active.

For the sake of simplicity from now on we will use $T(\mathbb{G})$ instead of $T(\mathbb{G}; x, y)$.

2.1. e is active in all spanning quasi-trees

First, observe the case when e is contained in all/none of the spanning quasi-trees. Moffatt shows with a resolution tree that it happens if and only if e is a bridge or a trivial loop in $\mathbb{G}$. A loop is called a trivial loop if there is no interlacing edge with it.

Notice that the two cases are dual to each other: if e is a bridge, then its dual is a trivial loop. Thus it is enough to prove the recursion for one of the cases, the other follows from $T(\mathbb{G}; x, y) = T(\mathbb{G}^*; y, x)$.

For a bridge e: $f(e) = 1$ and $g(e) = y - 1$ (since it is the same face on the two sides of the edge). For arbitrary spanning quasi-tree $\mathbb{T}$ by contracting e we obtain a spanning quasi-tree $\mathbb{T}/e$ since the sequence of edges generated by the tour does not change just the two instance of e is deleted. Thus in $\mathbb{T}/e$ an edge ($\neq e$) is active if and only if it was active in $\mathbb{T}$. Therefore we obtain a bijection between the spanning quasi-trees of $\mathbb{G}$ and $\mathbb{G}/e$ such that for every quasi-tree $ILO(\mathbb{T}/e) = ILO(\mathbb{T}) - 1$. Furthermore the number of vertices and edges decreases by one in $\mathbb{G}/e$ and the number of edges decreases by one in $\mathbb{T}/e$.

$$T(\mathbb{G}/e) = \sqrt{x-1}^{e(\mathbb{G})-1-b(\mathbb{G})+1} \cdot \sqrt{y-1}^{1-v(\mathbb{G})+1} \sum_{\mathbb{T}} \sqrt{\frac{y-1}{x-1}}^{e(\mathbb{T})-1} x^{ILO(\mathbb{T})-1} y^{ELO(\mathbb{T})}) =$$

$$= \sqrt{x-1}^{e(\mathbb{G})-b(\mathbb{G})+1} \cdot \sqrt{y-1}^{1-v(\mathbb{G})} \sum_{\mathbb{T}} \sqrt{\frac{y-1}{x-1}}^{e(\mathbb{T})} x^{ILO(\mathbb{T})-1} y^{ELO(\mathbb{T})}) = T(\mathbb{G})/x$$

If we delete e then $\mathbb{G}$ will no longer be connected, so we have to find a spanning quasi-tree in each component. Here we also have a bijection between the spanning quasi-trees in $\mathbb{G}$ and the spanning quasi-tree pairs in $\mathbb{G}\backslash e$. The ribbon graph polynomial of $\mathbb{G}\backslash e$ is the product of the polynomials of the two component. e has no interlacing edges so in the tour first we see all two instances of every edge from one of the components, then the two instances of the edges in the other component. Thus the activity of the remainding edges does not change. In $\mathbb{G}\backslash e$ the number of edges is one less than is $\mathbb{G}$, while examining the two components we count one of the faces twice and this way $b(\mathbb{G}\backslash e) = b(\mathbb{G}) + 1$. Now we can calculate the polynomial:

$$T(\mathbb{G}\backslash e) = T(\mathbb{G}_1) \cdot T(\mathbb{G}_2) =$$

$$= \left(\sqrt{x-1}^{e(\mathbb{G}_1)-b(\mathbb{G}_1)+1} \cdot \sqrt{y-1}^{1-v(\mathbb{G}_1)} \sum_{\mathbb{T}_1 \subseteq \mathbb{G}_1} \sqrt{\frac{y-1}{x-1}}^{e(\mathbb{T}_1)} x^{ILO(\mathbb{T}_1)} y^{ELO(\mathbb{T}_1)} \right) T(\mathbb{G}_2) =$$

$$= \sqrt{x-1}^{(e(\mathbb{G})-1)-(b(\mathbb{G})+1)+2} \cdot \sqrt{y-1}^{2-v(\mathbb{G})} \sum_{\mathbb{T}_1 \subseteq K_1} \sum_{\mathbb{T}_2 \subseteq K_2} \sqrt{\frac{y-1}{x-1}}^{e(\mathbb{T}_1)+e(\mathbb{T}_2)} x^{ILO(\mathbb{T}_1)+ILO(\mathbb{T}_2)} y^{ELO(\mathbb{T}_1)+ELO(\mathbb{T}_2)} =$$

$$= \frac{\sqrt{y-1}}{\sqrt{x-1}} \sqrt{x-1}^{e(\mathbb{G})-b(\mathbb{G})+1} \cdot \sqrt{y-1}^{1-v(\mathbb{G})} \sum_{\mathbb{T}} \sqrt{\frac{y-1}{x-1}}^{-1} \sqrt{\frac{y-1}{x-1}}^{e(\mathbb{T})} x^{ILO(\mathbb{T})-1} y^{ELO(\mathbb{T})} = T(\mathbb{G})/x$$

We can obtain the recursive formula from the two previous equations:

$$T(\mathbb{G}) = (x-1)T(\mathbb{G}\backslash e) + 1 \cdot T(\mathbb{G}/e) = (x-1)\frac{T(\mathbb{G})}{x} + \frac{T(\mathbb{G})}{x}$$

2.2. e is active in none of the spanning quasi-trees

We can assume that the edge e in the recursion is not active in any spanning quasi-trees. We will examine the cases depending on whether e or e^* is a loop.

Suppose that e is not a loop is $\mathbb{G}$ (in this case $g(e) = 1$).

- a1) If e^* is a loop in $\mathbb{G}^*$, then on the both side of e there is the same face in the embedded graph. Consider the spanning quasi-trees in $\mathbb{G}$ which contain e. Every such $\mathbb{T}_1$ corresponds to a different spanning quasi-tree $\mathbb{T}'$ in $\mathbb{G}/e$, and for every $\mathbb{T}'$ in $\mathbb{G}/e$ there is a unique $\mathbb{T}_1$ from which it can be obtained by contracting e. With this contraction the surface does not change, while the number of vertices and edges decreases in $\mathbb{G}$ and in $\mathbb{T}_1$ as well.

  $$T(\mathbb{G}/e) = \sqrt{x-1}^{(e(\mathbb{G})-1)-b(\mathbb{G})+1} \cdot \sqrt{y-1}^{1-(v(\mathbb{G})-1)} \sum_{\mathbb{T}_1 : e \in \mathbb{T}_1} \sqrt{\frac{y-1}{x-1}}^{e(\mathbb{T}_1)-1} x^{ILO(\mathbb{T}_1)} y^{ELO(\mathbb{T}_1)} =$$

  $$= \frac{\sqrt{y-1}}{\sqrt{x-1}} \sqrt{x-1}^{e(\mathbb{G})-b(\mathbb{G})+1} \cdot \sqrt{y-1}^{1-v(\mathbb{G})} \sum_{\mathbb{T}_1 : e \in \mathbb{T}_1} \sqrt{\frac{x-1}{y-1}} \sqrt{\frac{y-1}{x-1}}^{e(\mathbb{T}_1)} x^{ILO(\mathbb{T}_1)} y^{ELO(\mathbb{T}_1)} = T(\mathbb{G})|_{e \in \mathbb{T}_1}$$

  The spanning quasi-trees which does not contain e does not change when we delete the edge, but the surface into which $\mathbb{G}\backslash e$ can be embedded changes. Take the graph $\mathbb{G}$ cellularly embedded to surface Σ and look at the face F bordering e. Walking around F we get a sequence of edges (e appearing two times): $e, f, g, s, g, e, f, d...e$. After deleting e the walk splits into to parts, which means that there is a handle in the surface which is not used by any edges anymore. Thus $\mathbb{G}\backslash e$ can be cellularly embedded in Σ', where we obtained Σ' from Σ by removing the handle. This way after the deletion the number of edges decreased by one and the number of faces increased by one.

  $$f(e)T(\mathbb{G}\backslash e) = (x-1)T(\mathbb{G}\backslash e) =$$

  $$(x-1)\sqrt{x-1}^{(e(\mathbb{G})-1)-(b(\mathbb{G})+1)+1} \cdot \sqrt{y-1}^{1-v(\mathbb{G})} \sum_{\mathbb{T}_2 : e \notin \mathbb{T}_2} \sqrt{\frac{y-1}{x-1}}^{e(\mathbb{T}_2)} x^{ILO(\mathbb{T}_2)} y^{ELO(\mathbb{T}_2)} = T(\mathbb{G})|_{e \notin \mathbb{T}_2}$$

  In both cases the sequence of edges generated by the tour does not changes, except that e is deleted. e is the edge appearing first in the tour the the activity of the other edges does not change. For every edge, the set of interlacing edges can only change by e, but e did not effect any other edge.

  We proved that $T(\mathbb{G}) = T(\mathbb{G})|_{e \in \mathbb{T}_1} + T(\mathbb{G})|_{e \notin \mathbb{T}_2} = f(e)T(\mathbb{G}\backslash e) + g(e)T(\mathbb{G}/e)$ for an edge e such that e is not a loop and e^* is a loop.

- b) If e^* is not a loop, then e separates two different faces of the embedded graph. There is a bijection between the spanning quasi-trees $\mathbb{T}_1$ containing e and the spanning quasi-trees of the graph obtained by contracting e. With this operation the surface does not changes, while the number of edges and vertices decreases by one in the graph and the spanning quasi-trees as well. Taking the tour of $\mathbb{T}_1$ the sequence of seen edges only changes by the disappearance of e (the order stays the same), so the activity of the remaining edges does not change.

  $$T(\mathbb{G}/e) = \sqrt{x-1}^{e(\mathbb{G})-1-b(\mathbb{G})+1} \cdot \sqrt{y-1}^{1-v(\mathbb{G})+1} \sum_{\mathbb{T}_1 : e \in \mathbb{T}_1} \sqrt{\frac{y-1}{x-1}}^{e(\mathbb{T}_1)-1} x^{ILO(\mathbb{T}_1)} y^{ELO(\mathbb{T}_1)} = T(\mathbb{G})|_{e \in \mathbb{T}_1}$$

Every spanning quasi-tree not containing e is a spanning quasi-tree in $\mathbb{G}\backslash e$ as well. With deleting e the number of edges and faces of the graph decreases by one (since we merge the two faces bordering e). In this case the order of the edges in the tour does not change, so the nuber of active edges is the same after the deletion.

$$T(\mathbb{G}\backslash e)=\sqrt{x-1}^{e(\mathbb{G})-1-(b(\mathbb{G})-1)+1}\cdot\sqrt{y-1}^{1-v(\mathbb{G})}\sum_{\mathbb{T}_2:e\notin\mathbb{T}_2}\sqrt{\frac{y-1}{x-1}}^{e(\mathbb{T}_2)}x^{ILO(\mathbb{T}_2)}y^{ELO(\mathbb{T}_2)}=T(\mathbb{G})|_{e\notin\mathbb{T}_2}$$

We obtained that the recursion works, when neither e, nor e^* is a loop:

$$T(\mathbb{G})=T(\mathbb{G})|_{e\in\mathbb{T}_1}+T(\mathbb{G})|_{e\notin\mathbb{T}_2}=T(\mathbb{G}\backslash e)+T(\mathbb{G}/e)$$

We showed that the recursion works for non-loop edge e, where the dual edge e^* is a loop. Because of the duality if e is a loop and e^* is not a loop then the same proof works:

- a2) The spanning quasi-trees containing e have one-to-one correspondence to a spanning quasi-tree in $\mathbb{G}/e$. The surface in which $\mathbb{G}/e$ can be cellularly embedded differs from the one for $\mathbb{G}$. When contracting e we make two vertices instead of the one (endpoint of e): partitioning the incident edges in two based on how e splits circular order of the edges leaving the vertex in two part. This way the order of the edges in the tour of $\mathbb{T}_1$ does not change (except e disappearing), so the activities does not change. The number of edges in the graph and the $\mathbb{T}_1$ spanning tree decreases by one and the number of vertices of $\mathbb{G}$ increases by one.

  $$g(e)T(\mathbb{G}/e)=(y-1)T(\mathbb{G}/e)=$$

  $$(y-1)\sqrt{x-1}^{(e(\mathbb{G})-1)-b(\mathbb{G})+1}\cdot\sqrt{y-1}^{1-(v(\mathbb{G})+1)}\sum_{\mathbb{T}_1:e\in\mathbb{T}_1}\sqrt{\frac{y-1}{x-1}}^{e(\mathbb{T}_1)-1}x^{ILO(\mathbb{T}_1)}y^{ELO(\mathbb{T}_1)}=T(\mathbb{G})|_{e\in\mathbb{T}_1}$$

  The spanning quasi-trees not containing e will be spanning quasi-trees in $\mathbb{G}\backslash e$ as well. By the deletion the surface does not change and the number of edges and faces decreases by one.

  $$T(\mathbb{G}\backslash e)=\sqrt{x-1}^{(e(\mathbb{G})-1)-(b(\mathbb{G})-1)+1}\cdot\sqrt{y-1}^{1-v(\mathbb{G})}\sum_{\mathbb{T}_2:e\notin\mathbb{T}_2}\sqrt{\frac{y-1}{x-1}}^{e(\mathbb{T}_2)}x^{ILO(\mathbb{T}_2)}y^{ELO(\mathbb{T}_2)}=T(\mathbb{G})|_{e\notin\mathbb{T}_2}$$

  We proved that $T(\mathbb{G})=T(\mathbb{G})|_{e\in\mathbb{T}_1}+T(\mathbb{G})|_{e\notin\mathbb{T}_2}=f(e)T(\mathbb{G}\backslash e)+g(e)T(\mathbb{G}/e)$

- c) The last case is when both e and e^* are loops. In this case the surface changes if we contract or delete e.

  The spanning quasi-trees containing e are in bijection with the spanning quasi-trees of $\mathbb{G}/e$. By contracting e the number of edges decreases by one in the graph and the spanning quasi-trees as well, while the number of vertices increase by one. (The surface changes the same way that it did in the a2) part). The activities of the edges does not change since the order of the edges in the tour stays the same.

  $$g(e)T(\mathbb{G}/e)=(y-1)T(\mathbb{G}/e)=$$

  $$(y-1)\sqrt{x-1}^{(e(\mathbb{G})-1)-b(\mathbb{G})+1}\cdot\sqrt{y-1}^{1-(v(\mathbb{G})+1)}\sum_{\mathbb{T}_1:e\in\mathbb{T}_1}\sqrt{\frac{y-1}{x-1}}^{e(\mathbb{T}_1)-1}x^{ILO(\mathbb{T}_1)}y^{ELO(\mathbb{T}_1)}=T(\mathbb{G})|_{e\in\mathbb{T}_1}$$

  The spanning quasi-trees not containing e are also spanning quasi-trees in $\mathbb{G}\backslash e$. The surface changes the way we described in a1): with the deletion the number of edges decreased by one and the number of faces increased by one.

  $$f(e)T(\mathbb{G}\backslash e)=(x-1)T(\mathbb{G}\backslash e)=$$

  $$(x-1)\sqrt{x-1}^{(e(\mathbb{G})-1)-(b(\mathbb{G})+1)+1}\cdot\sqrt{y-1}^{1-v(\mathbb{G})}\sum_{\mathbb{T}_2:e\notin\mathbb{T}_2}\sqrt{\frac{y-1}{x-1}}^{e(\mathbb{T}_2)}x^{ILO(\mathbb{T}_2)}y^{ELO(\mathbb{T}_2)}=T(\mathbb{G})|_{e\notin\mathbb{T}_2}$$

  Merging the two equations we proved that $T(\mathbb{G})=T(\mathbb{G})|_{e\in\mathbb{T}_1}+T(\mathbb{G})|_{e\notin\mathbb{T}_2}=f(e)T(\mathbb{G}\backslash e)+g(e)T(\mathbb{G}/e)$.

Bernardi's approach of defining the activities works for the ribbon graph polynomial.

References

[1] Olivier Bernardi. “A Characterization of the Tutte Polynomial via Combinatorial Embeddings”. *arXiv preprint math/0608057* (2006). arXiv: `math/0608057 [math.CO]`. URL: `https://arxiv.org/abs/math/0608057`.

[2] Péter Csikvári. *The Tutte Polynomial – Lecture Notes*. Lecture notes. n.d. URL: `https://csikvarip.web.elte.hu/Tutte_polynomial_lecture_note.pdf`.

[3] Joanna A. Ellis-Monaghan és Criel Merino. “Graph Polynomials and Their Applications I: The Tutte Polynomial”. *arXiv preprint arXiv:0803.3079* (2008). arXiv: `0803.3079 [math.CO]`. URL: `https://arxiv.org/abs/0803.3079`.

[4] Iain Moffatt. “Constructing a Tutte Polynomial for Graphs Embedded in Surfaces”. *arXiv preprint arXiv:2502.15318* (2025). arXiv: `2502.15318 [math.CO]`. URL: `https://arxiv.org/abs/2502.15318`.