Directed studies

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1 Introduction

The following is a short introduction of the prerequisites to the discussion of the Milnor fibration, its consequences and topological and algebraic invariants of analytic set germs, relying on [2]. Later on we use $\mathbb{K}$, where either $\mathbb{R}$ or $\mathbb{C}$ would be admissible.

Definition 1.1. We call $V \subset \mathbb{K}^n$ an algebraic set, if it is the common zero set of a collection of $f_k \in \mathbb{K}[x_1, \ldots, x_n]$ polynomials.

Such a set V is called irreducible or algebraic variety if it cannot be decomposed into algebraic subsets, that is for any $\varnothing \neq V_1, V_2 \subsetneq V$ algebraic subsets $V_1 \cup V_2 \neq V$.
Obviously there are multiple set of polynomials that have the same common zero set ($\{f\}$ and $\{2f\}$ for example). However it is not too difficult to see, that algebraic sets in $\mathbb{K}^n$ are uniquely matched by radical ideals in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$. Described in detail in chapter 1. corollary 1.4 of [3] Let $I(V)$ denote this ideal for an algebraic set V, and $V(I)$ its inverse.
A natural property of such geometric objects, is their dimension. And indeed it can be formalized in the following way:

Definition 1.2. Let V be an algebraic set. Then for a longest $V \supsetneq V_1 \supsetneq \cdots \supsetneq V_m$ chain where V_j are non-empty algebraic varieties, m is the dimension of V.

First of all $m < \infty$, since Hilbert's basis theorem asserts, that every ideal in $\mathbb{K}[x_1, \ldots, x_n]$ is spanned by finitely many polynomials. There are several equivalent definitions for the dimension of an algebraic set (such as the Krull dimension of commutative rings), with this being arguably the most intuitive.
Let us now consider an algebraic set V, and let $\{f_1, \ldots, f_k\}$ span $I(V)$.

Definition 1.3. A point $x \in V$ is called non-singular if the matrix $(\partial_j f_i(x))_{ij}$ attains full rank (as in maximal for x), and singular if not.

Remark. This definition might produce unintuitive outcomes when dealing with reducible algebraic sets, where components have different dimensions. Example being the union of a line and a point in $\mathbb{C}^2$, in which case the only the point is regular.

Example 1.4. Let

$$f(x, y) = x^2 + xy + y^3$$

When we restrict $V(f)$ to $\mathbb{R}^2$ we see a double point at the origin. And also $\nabla f(x, y) = (2x + y, 3y^2 + x)$, meaning the above-described matrix (here just a vector) has rank 0 instead of the maximal 1 at $(0,0)$. Therefore $(0,0)$ is a singular point or a singularity of $V(f) \subset \mathbb{C}^2$. We can also deduce, that there is one other singular point $\left(-\frac{1}{12}, \frac{1}{6}\right)$, since $\nabla f \left(-\frac{1}{12}, \frac{1}{6}\right) = (0,0)$ (which the graphing program does not even mark as a point of $V(f)$, but nonetheless is).

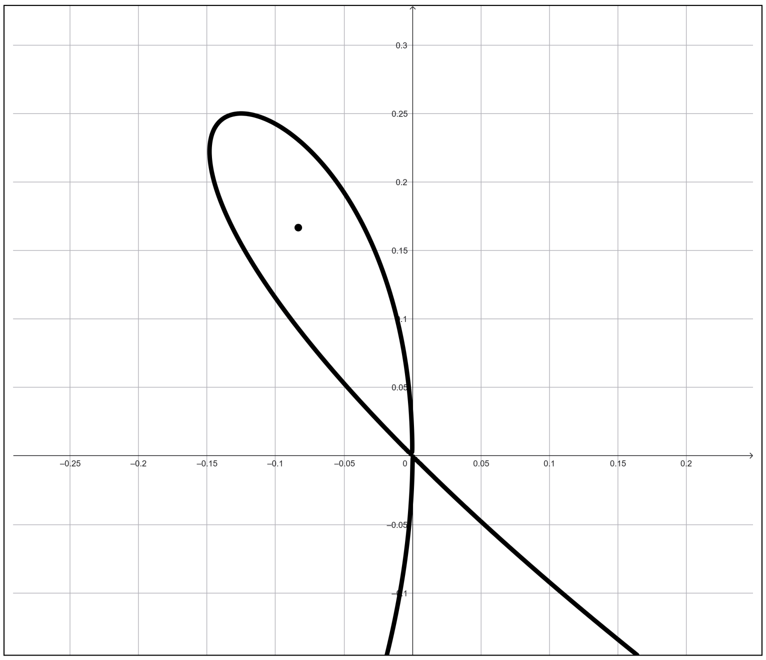

Figure 1: $V(f)$ restricted to $\mathbb{R}^2$, with the second singular point $\left(-\frac{1}{12}, \frac{1}{6}\right)$ added manually.

Our way of examining these sets will be to take a small sphere centered around an isolated singularity (or any of its regular points for that matter) and take its intersection with the algebraic set in question. Such an intersection will be referred to as the link of a singularity. Since we are interested in local qualities, it is useful to introduce some definitions.

Definition 1.5. An algebraic set germ around $x \in \mathbb{K}^n$ is an equivalence class of algebraic sets that contain x and are pairwise equivalent on some neighborhood of x. That is to say $V \sim W$ if $x \in V, x \in W$ and there exists some $x \in U \subset \mathbb{K}^n$ neighborhood, such that $U \cap V = U \cap W$.

Such an equivalence class can analogously be defined on analytic functions, analytic sets (which are defined similarly to algebraic sets) and numerous other mathematical objects, however we will only need the three specified. For clarity's sake:

Definition 1.6. An analytic function germ around $x \in \mathbb{K}^n$ is an equivalence class of analytic $\mathbb{K}^n \to \mathbb{K}$ functions that have x as their common zero, and in which for any pair of functions f, g there exists a $x \in U \subset \mathbb{K}^n$ neighborhood, such that $f|_U = g|_U$.

These germs form a noetherian local ring with addition and multiplication inherited from analytic functions. Note also that $\mathbb{K}^n$ can be replaced with any $x \in X \subset \mathbb{K}^n$ analytic set. Such rings are usually denoted $\mathcal{O}_{\mathbb{K}^n,x}$ and $\mathcal{O}_{X,x}$ respectively.

We now look at what happens near an isolated singularity. (Meaning it is the only singular point of the algebraic set in some neighborhood.)

Proposition 1.7. *Let $x \in V$ be a regular point or an isolated singularity of an algebraic set. Then $S_\varepsilon \cap V$ ($S_\varepsilon = \{y \in \mathbb{K}^n : \|y - x\| = \varepsilon\}$) is a smooth manifold for sufficiently small ε.*

At first this might seem arbitrary, but the following theorem gives us insight into the reasoning behind such analysis.

Theorem 1.8. *Given a $V \subset \mathbb{K}^n$ algebraic set, and $x \in V$ isolated singularity or a regular point. Then for a small enough ε, the cone over $S_\varepsilon \cap V$ with base point x is equivalent in the following sense to $D_\varepsilon \cap V$:*

$$(D_\varepsilon, \{x + t(y - x) \in \mathbb{K}^n : \forall t \in [0, 1], \forall y \in S_\varepsilon \cap V\}) = (D_\varepsilon, Cone(S_\varepsilon \cap V)) \simeq (D_\varepsilon, D_\varepsilon \cap V)$$

Both the proposition and theorem have proofs that rely on the following often useful lemma:

Lemma 1.9. *Let $V \subset \mathbb{R}^n$ be an algebraic set, and U a set defined as such:*

$$U = \{x \in \mathbb{R}^n | g_1(x) > 0, \dots, g_m(x) > 0\},$$

where g_i are polynomials. If $U \cap V$ has points arbitrarily close to the origin, then there exists a real analytic curve, $p : [0, \varepsilon) \to \mathbb{R}^n$, with $p(0) = 0$ and $p(t) \in V \cap U$ for $t \in (0, \varepsilon)$.

The equivalence in theorem 1.8 is referred to as embedded topological equivalence, since it is not only sensitive to the topology of V, but also its embedding into $\mathbb{K}^n$ locally around x. Using the language of germ we can say, that an algebraic set germ around x (where x is either an isolated singularity or a regular point) defines a cone over a link which is in embedded topological equivalence with all members of the equivalence class. Therefore this assignment of the cone to the germ is meaningful. As we might assume from the example earlier, algebraic sets germs around a regular point offer little to examine topologically speaking.

Proposition 1.10. *Let x be a regular point of V an algebraic set. Then $S_\varepsilon \cap V$ is an unknotted sphere in S_ε.*

The following is a small tangent on algebraic curves. (Detailed explanations can be found in section 8. of [1].)

2 Special case of the complex plane and algebraic curves

In the case of algebraic curves (which are also surfaces as they have real dimension 2) in the two dimensional complex space, we have a nice characterization. In this case the link of a singularity is a one dimensional closed smooth manifold (a disjunct union of circles). Let us look at an example.

Example 2.1. Let

$$f(x, y) = x^3 - y^2.$$

The solutions for $f(x,y)=0$ are $y=\epsilon x^{\frac{3}{2}}$, where $\epsilon^2=1$. If we restrict the solutions to a small sphere around $(0,0)$, or equivalently to the border of $D^2_{\varepsilon_1}\times D^2_{\varepsilon_2}$, with $\varepsilon_1^3=\varepsilon_2^2$ we have the following.

$$\partial(D^2_{\varepsilon_1}\times D^2_{\varepsilon_2})=\partial D^2_{\varepsilon_1}\times D^2_{\varepsilon_2}\cup_{\partial D^2_{\varepsilon_1}\times\partial D^2_{\varepsilon_2}} D^2_{\varepsilon_1}\times\partial D^2_{\varepsilon_2}\simeq S^1_{\epsilon_1}\times D^2_{\varepsilon_2}\cup_{S^1_{\varepsilon_1}\times S^1_{\varepsilon_2}} D^2_{\varepsilon_1}\times S^1_{\varepsilon_2}$$

Which is two solid 2-tori glued together on their borders. If we restrict the solutions to this border we have that either $|x|=\varepsilon_1$ or $|y|=\varepsilon_2$, but since $x^3=y^2$, $|x|=\varepsilon_1\Leftrightarrow|y|=\varepsilon_2$. Meaning $V(f)\cap D^2_{\varepsilon_1}\times D^2_{\varepsilon_2}\subset S^1_{\varepsilon_1}\times S^1_{\varepsilon_2}$. These solutions on the torus obviously have the form $\left(\varepsilon_1 e^{2it},\varepsilon_2 e^{3it}\right)$ where $0\leq t\leq 2\pi$. Plotted on the torus we see a trefoil knot or a toric knot of type $(2,3)$.

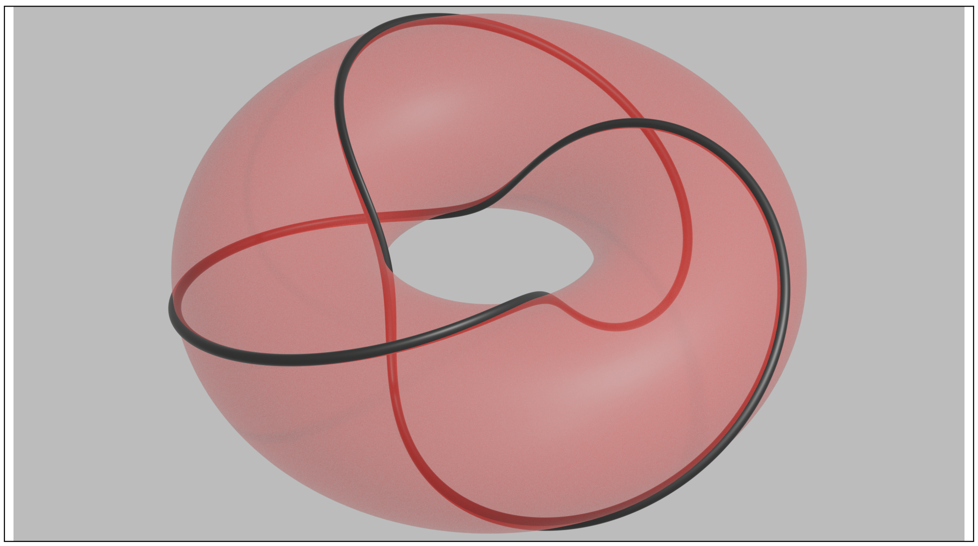

Figure 2: Toric knot of type $(2,3)$ (trefoil knot) on the torus

More generally if $f(x,y)=x^q-y^p$, we can say that if d is the greatest common divisor of q and p then the link of the singularity is d copies of a toric knot of type $(\frac{p}{d},\frac{q}{d})$ interlinked, where the linking number is determined by the so-called intersection multiplicity of the respective components.
The complete characterization is as follows:

Theorem 2.2. *Let (V_1,x), and (V_2,y) be algebraic curve germs in $\mathbb{C}^2$. (That is to say that (V_1,x) is the equivalence class of algebraic curves around x defined by $V_1\ni x$.) Then (V_1,x) is in embedded topological equivalence with (V_2,y) if and only if there is a one to one correspondence between their irreducible components such that:*

1. *the (iterated toric) knot of the respective components is equivalent,*
2. *the linking number of the respective knots are in agreement.*

3 Milnor fibration

So far we have used the link of a singularity to characterize (in the case of complex algebraic plane curves completely characterize) the algebraic sets locally in $\mathbb{K}^n$. However there is a result of Milnor that states the following:

Theorem 3.1. *Let $V(f) \in \mathbb{C}^n$ be a hypersurface, with an isolated singularity z. Then for a small enough ε, with $K = S_\varepsilon \cap V(f)$ (the link of the isolated singular point z) $S_\epsilon \setminus K$ has a locally trivial fibration over S^1 with*

$$\Phi : S_\varepsilon \setminus K \to S^1, \quad \Phi(z) = \frac{f(z)}{|f(z)|}$$

as projection map.

Such a fiber bundle has smooth fibers $F_\theta = \Phi^{-1}(e^{i\theta})$, where each of them has the link as their boundary, $\partial F_\theta = K$, and fit around K in manned that is similar to the pages of an open book around its spine (when looking at cases of lower dimension). Hence a situation as this is referred to as an open book decomposition of $S_\varepsilon \simeq S^{2n-1}$.

Example 3.2. First let

$$f(x, y) = x.$$

In this case every point is non-singular, in particular $(0, 0)$ is regular. Hence we know, that the link associated with $(0, 0)$ is an unknotted S^1. (Even though the stated theorem does not apply, the open book decomposition exists.) In this case $\Phi(x, y) = \frac{x}{|x|}$, meaning for a $e^{i\theta} \in S^1$

$$F_\theta = \Phi^{-1}(e^{i\theta}) = \{(x, y) \in S_\epsilon \setminus S^1 |\, \arg(x) = \theta\} \simeq D^2$$

We can use a stereographic projection to visualize this in $\mathbb{R}^3$.

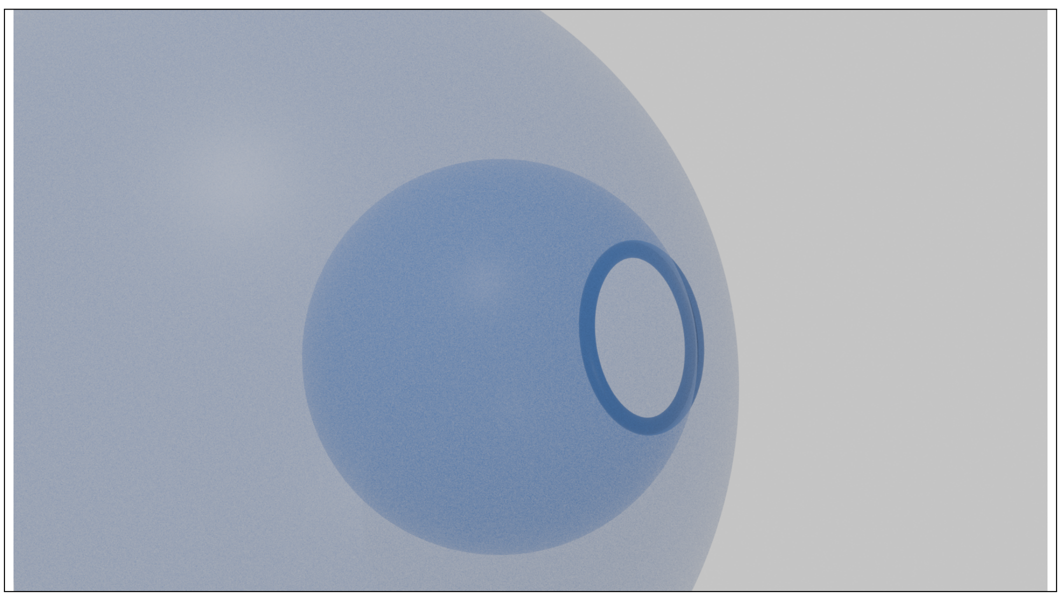

Figure 3: The base space S^1, and two fibers in $\mathbb{R}^3$, stereographically projected form S^3 with point of projection $(1, 0, 0, 0)$

Example 3.3. Now let us take

$$f(x, y) = xy.$$

$V(f)$ has an isolated (solitary) singular point in $(0, 0)$. The link

$$K = S^3 \cap V(f) = \{(x, y) \in \mathbb{C}^2 | |x|^2 + |y|^2 = 1, xy = 0\} = \\ = \{(x, y) \in \mathbb{C}^2 | x = 0, |y| = 1\} \cup \{(x, y) \in \mathbb{C}^2 | |x| = 1, |y| = 0\}$$

Which is two interlinked circles in S^3, also known as a Hopf link. The projection $\Phi : S^3 \setminus K \to S^1$ now is $\Phi(x, y) = \frac{xy}{|xy|}$ and

$$\Phi^{-1}(e^{i\theta}) = \{(x, y) \in S^3 | \arg(x) + \arg(y) \equiv \theta \mod 2\pi\} \simeq S^1 \times [0, 1].$$

That is to say that each fiber is homeomorphic to a cylinder without its base and top.

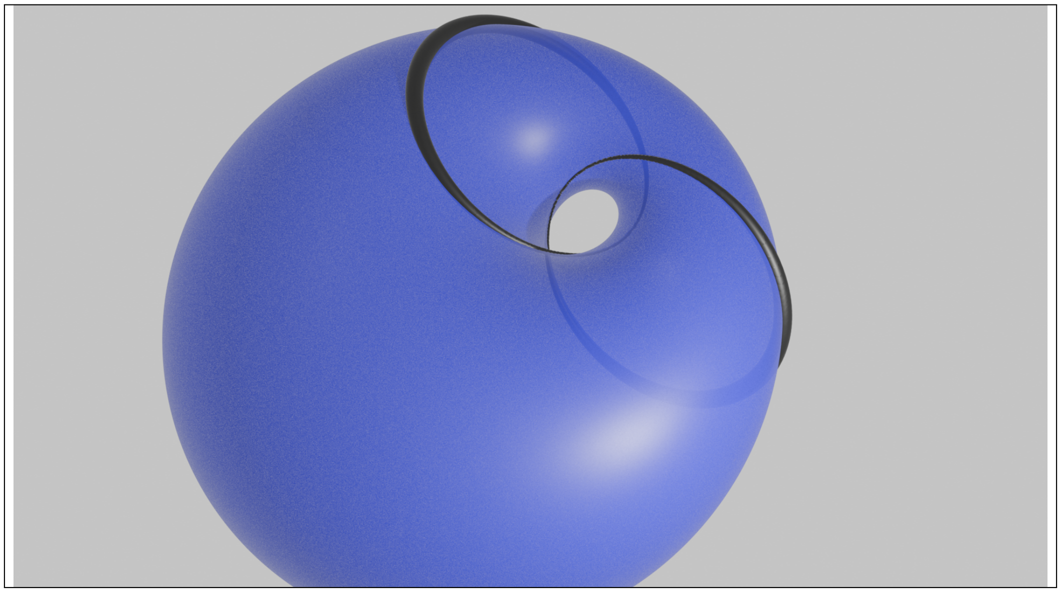

Figure 4: The Hopf link together with two fibers ($\theta = \frac{\pi}{4}, \frac{5\pi}{4}$). The non-constant thickness of the link is quirk of the visualization, it is actually the set $|x \cdot y| = 0.03$, which approximates the link closely. Again stereographic projection was used, with point of projection $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

References

[1] Brieskorn, E, Knörrer, H.: *Plane Algabreic Curves*, Birkhäuser, 1986

[2] Milnor, J.: *Singular Points of Complex Hypersurfaces*, Princeton Univerity Press, 1968

[3] Hartshorne, R.: *Algebraic Geometry*, Graduate Texts in Mathematics **52**, Springer, 1977