Vertex matroid families

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Introduction

We aim to investigate vertex matroid families motivated by Garamvölgyi [1], who showed interesting properties of graph matroid families. We use the notations of [2], to which we refer the reader for the relevant definitions and notation.

Definition 1. *A* ***graph matroid family*** $\mathcal{M}$ *is a family of matroids, defined on the edge set of each (finite, simple) graph in a way that is*

- *well defined: every graph isomorphism* $\varphi : V(G) \to V(H)$ *induces an isomorphism between* $\mathcal{M}(G)$ *and* $\mathcal{M}(H)$.
- *compatible: for all subgraph* H *of* G, $\mathcal{M}(H)$ *is a restriction of* $\mathcal{M}(G)$.

These families show interesting structure. For example, if we take any graph matroid family $\mathcal{M}$ and look at the matroids $\mathcal{M}(K_n)$ for all $n \in \mathbb{N}$ their ranks are linear for $n \geq t$ for some threshold t.

Garamvölgyi's work suggests that studying families of graph-based matroids can be highly productive. Several well-known matroid families—such as matching matroids, transversal matroids, and gammoids—are defined on the vertex sets of finite graphs. This naturally motivates exploring whether one can define an analogous family of vertex matroids in the same spirit as these graph matroid families.

Definition 2. *A* ***vertex matroid family*** $\mathcal{M}$ *is a family of matroids, defined on the vertex set of each (finite, simple) graph in a way that is*

- *well defined: every graph isomorphism* $\varphi : V(G) \to V(H)$ *induces an isomorphism between* $\mathcal{M}(G)$ *and* $\mathcal{M}(H)$.
- *compatible: for all induced subgraph* H *of* G, $\mathcal{M}(H)$ *is a restriction of* $\mathcal{M}(G)$.

This natural definition sadly does not apply to the above mentioned matroid families. A simple counterexample is the matching matroid of the graph consisting of two vertices and an edge between them. If we restrict the matching matroid of this graph to a single vertex, we obtain the free matroid on one element. However, if we delete a vertex from the graph and then take the matching matroid of the resulting graph, the corresponding element becomes a loop. Still, it is a natural question to try to characterize vertex matroid families.

Let $G = (V, E)$ be a graph and H be the subgraph of G spanned by $V' \subset V$. Let $\mathcal{M}$ be a vertex matroid family. If V' is an independent set in $\mathcal{M}(G)$, then $\mathcal{M}(G)|_{V'} = \mathcal{M}(H)$ is the free matroid on $|V'|$ elements. Intuitively, if H has some sort of property in $\mathcal{M}(G)$ in the matroid sense then it is inherited to $\mathcal{M}(H)$. This motivates to introduce the following definitions:

Definition 3. *Let* $\mathcal{M}$ *be a vertex matroid family. We call* $G = (V, E)$ *an* $\mathcal{M}$*-independent set /* $\mathcal{M}$*-dependent set /* $\mathcal{M}$*-circuit, if* V *is independent / dependent / a circuit in* $\mathcal{M}(G)$.

Two simple examples for vertex matroid families are the free matroid family that assigns to each graph G the free matroid on its vertices, and the k-uniform matroid family which assigns to each graph G the $\min\{k, |V(G)|\}$-uniform matroid on its vertices for each graph $G = (V, E)$. We show these are all the vertex matroid families:

Theorem 1. *For all vertex matroid family* $\mathcal{M}$ *either all matroids are free, or there exists an integer* k *such that for all finite simple graphs* G, $\mathcal{M}(G)$ *is the uniform matroid on* $|V|$ *elements with rank* $\min\{|V|, k\}$.

Proof of Theorem 1

First, we define an operation that will be a useful tool for proving Theorem 1. Intuitively, we construct the graph by gluing together two n-vertex graphs along $n - 1$ shared vertices, with the two unpaired vertices remaining separate and retaining all of their original neighbors.

Definition 4. *(gluing operation) Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ and $v_1 \in V_1$, $v_2 \in V_2$ be such that for the induced graphs $G_1[V_1 - v_1] \cong G_2[V_2 - v_2] \cong G'$ for some graph $G' = (V', E')$, where $\cong$ denotes that they are isomorphic. Let $\varphi_i : V_i \to V'$ be a function that induces a graph isomorphism between G_i and G', $i = 1, 2$. The gluing of G_1 and G_2 along V' with isomorphisms φ_i $(i = 1, 2)$ is the graph consisting of G', and two vertices u_1 and u_2. For all neighbors w of v_i there is an edge between u_i and $\varphi_i(w)$, $i = 1, 2$. We call v_1 and v_2 unglued vertices. We denote the gluing of G_1 and G_2 along V' with isomorphisms φ_i $(i = 1, 2)$, having $u_1 \in V_1$ and $u_2 \in V_2$ as unglued vertices with $K(G_1, G_2, V', \varphi_1, \varphi_2)$. Later, when the context makes it clear, we omit certain parameters for simplicity and denote the gluing $K(G_1, G_2)$. With a slight abuse of notation, later we will identify the vertices v_1 and v_2 with u_1 and u_2, respectively.*

The following claim offers a convenient way to construct new dependent sets from known circuits using the gluing operation.

Claim 1. *Assume that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are $\mathcal{M}$-circuits for some vertex matroid family $\mathcal{M}$ and we can apply the gluing operation on them along some V' with some isomorphisms φ_1, φ_2, and unglued vertices $v_1 \in V_1$ and $v_2 \in V_2$. Then for all $u \in V'$, $K(G_1, G_2) - u$ is $\mathcal{M}$-dependent.*

Proof. We know, that V_1 in $\mathcal{M}(G_1)$ and V_2 in $\mathcal{M}(G_2)$ are circuits. In $\mathcal{M}(K(G_1, G_2)$ the graph induced by $V' \cup v_i$ is isomorphic with G_i for $i = 1, 2$ and u is in both of them. We can apply the circuit elimination axiom for u, that is, there exists a circuit $C \subseteq (V' \cup v_1 \cup v_2) - u$ in $\mathcal{M}(K(G_1, G_2))$. □

Notice that adding the edge v_1v_2 does not affect the correctness of the proof of the claim above, so when we use the gluing operation, we may add this edge.

Assume that we are given a vertex matroid family $\mathcal{M}$ for which $\mathcal{M}(G)$ is not the free matroid for all graph G, and that the smallest circuit is of size n. Let G_1 and G_2 be $\mathcal{M}$-circuits of size n and assume that we can apply the gluing operation on them. Then for any $u \in V'$, the graph $K(G_1, G_2) - u$ is an $\mathcal{M}$-circuit as it is dependent by Claim 1 and is of size n.

Claim 2. *Let $\mathcal{M}$ be a vertex matroid family. Assume that the smallest $\mathcal{M}$-circuit is on n vertices. Then the empty graph on n vertices is an $\mathcal{M}$-circuit.*

Proof. Let G be the $\mathcal{M}$-circuit with the most number of isolated vertices in it and among these graphs having smallest minimal degree apart from the isolated vertices. For the sake of contradiction, assume that G is not the empty graph. Choose a vertex $v \in V$ with minimal non-zero degree and one of its neighbors u. Let $G_0 = (V_0, E_0) \cong G$ and let $\varphi : G \to G_0$ be a graph isomorphism. Then apply the gluing operation on G and G_0 with having v and $\varphi(v)$ as unglued vertices, and receive $K(G, G_0)$, for some isomorphisms φ_1 and φ_2 between graphs $G - v, G_0 - \varphi(v)$ and some graph $G' = (V', E')$, respectively, which exists, as $G - v \cong G_0 - \varphi(v)$. We know by Claim 1 that $K(G, G_0) - \varphi_1(u)$ is an $\mathcal{M}$-circuit. In $K(G, G_0) - \varphi_1(u)$ the unglued vertices have smaller degree, than v in G. This means that they have lower non-zero degree, or that they are isolated. Both cases lead to contradiction by the choice of G. □

Now we are ready to prove that for a vertex matroid family $\mathcal{M}$ if its smallest $\mathcal{M}$-circuit is on n vertices, then all graphs on n vertices are $\mathcal{M}$-circuits.

Proof by induction on the number of edges in the $\mathcal{M}$-circuits. By Claim 2, the empty graph on n vertices is an $\mathcal{M}$-circuit. Assume that all graphs on n vertices with less than k edges are $\mathcal{M}$-circuits. We show that any graph $G = (V, E)$ on n vertices and k edges are $\mathcal{M}$-circuits. Let $v_1v_2 \in E$, $v_1, v_2 \in V$. Define $G_1 = (V - v_1 + u, E_1)$, $G_2 = (V - v_2 + u, E_2)$, where E_i is obtained by removing the edges on v_i from E, $i = 1, 2$, and u is an isolated vertex. Note that G_1 and G_2 have less than k edges, hence they are $\mathcal{M}$-circuits by induction. Now, apply the gluing operation on G_1 and G_2 along $V' = V - v_1 - v_2 + u$ (with the identity isomorphisms) and receive $K(G_1, G_2)$. G is the graph obtained from $K(G_1, G_2) - u$ by adding the edge v_1v_2, hence by the comment after Claim 1, that we may add an edge between the two unglued edges, we obtain that G is indeed an $\mathcal{M}$-circuit. The proof of Theorem 1 follows

References

[1] Dániel Garamvölgyi. Rigidity and reconstruction in matroids of highly connected graphs, 2024.

[2] James G Oxley. *Matroid theory*, volume 3. Oxford University Press, USA, 2006.