

Linear extensions of partially ordered sets

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Introduction

- ▶ Given a set P and a partial ordering $\prec$ on it, let us count the number of linear extensions of it
- ▶ This is the number of topological orderings of the *cover graph* $G = (V, A)$ of P , where $V = P$ and $A = \{uv \in P^2 : u \prec v, \nexists w \in P \setminus \{u, v\} : u \prec w \prec v\}$

Previous results

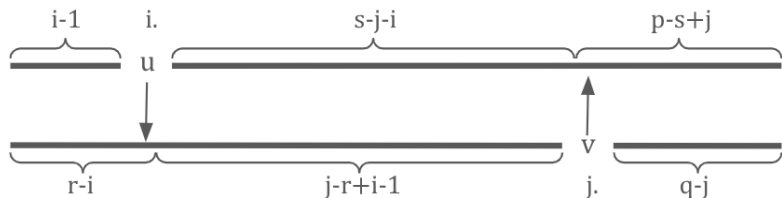
- ▶ G. Brightwell and P. Winkler: the problem is $\#P$ -hard [2]
- ▶ Kangas et al.: dynamic programming-based algorithm with time complexity of $O(n^{t+4})$ [3]
- ▶ K. Kangas, M. Koivisto and S. Salonen: improved to $O(n^{t+3})$ using nice tree decomposition [4]

An interesting special case

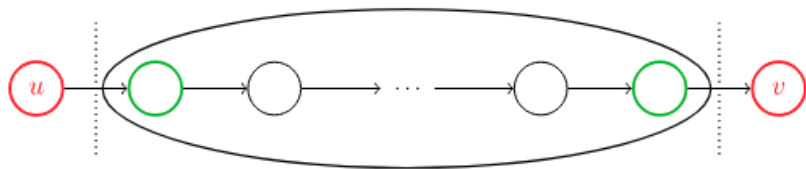
- ▶ When the cover graph is a tree, there exists an algorithm with a running time of $O(n^2)$, given by M. D. Atkinson [1]
- ▶ *Spectrum*: $\sigma_G : V \rightarrow \mathbb{Z}_+^n$
- ▶ $\sigma_G(v)_i$ is the number of topological orders in which the vertex v is in the i -th position
- ▶ Let uv be an arc, and split up the graph along this arc. Determine the spectrum of u and v recursively, and then join them together

Common spectrum, arc spectrum

- ▶ *Common spectrum*: $\tau_G : V^2 \rightarrow \mathbb{Z}_+^{n \times n}$
- ▶ $\tau_G(u, v)_{ij}$ is the number of topological orders in which the vertex u is in the i -th position, vertex v is in the j -th position
- ▶ Special cases solved: the two vertices are connected with an arc (*arc spectrum*) or with a directed path



Common spectrum, arc spectrum



Future plans

- ▶ Calculating the common spectrum of two vertices which are not connected by a directed path
- ▶ Calculating the common spectrum of 3, 4 or arbitrary k vertices

References



M. D. Atkinson.

On computing the number of linear extensions of a tree.
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G. Brightwell and P. Winkler.

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In *Proceedings of the Twenty-Third Annual ACM Symposium on Theory of Computing*, pages 175–181, 1991.



K. Kangas, T. Hankala, T. M. Niinimäki, and M. Koivisto.

Counting linear extensions of sparse posets.
In *Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence*, pages 603–609, 2016.



K. Kangas, M. Koivisto, and S. Salonen.

A faster tree-decomposition based algorithm for counting linear extensions.
Algorithmica, 82(8):2156–2173, 2020.

Usage of AI tools

- ▶ Enhance the code (mostly in logging)
- ▶ Writing small helper scripts in Python (e.g. drawing a tree based on it's edge list)