Linear extensions of partially ordered sets

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Introduction

- Given a set P and a partial ordering $\prec$ on it, let us count the number of linear extensions of it
- This is the number of topological orderings of the *cover graph* $G = (V, A)$ of P, where $V = P$ and $A = \{uv \in P^2 : u \prec v, \nexists w \in P \setminus \{u, v\} : u \prec w \prec v\}$

Previous results

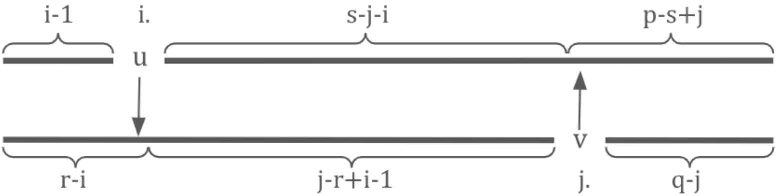
- G. Brightwell and P. Winkler: the problem is $\#P$-hard.[2]
- Kangas et al.: dynamic programming-based algorithm with time complexity of $O(n^{t+4})$[3]
- K. Kangas, M. Koivisto and S. Salonen: improved to $O(n^{t+3})$ using nice tree decomposition[4]

An interesting special case

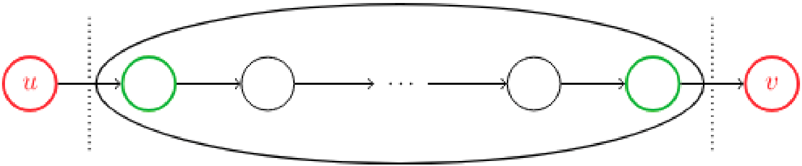
- When the cover graph is a tree, there exists an algorithm with a running time of $O(n^2)$, given by M. D. Atkinson. [1]
- *Spectrum*: $\sigma_G : V \to \mathbb{Z}_+^n$
- $\sigma_G(v)_i$ is the number of topological orders in which the vertex v is in the i-th position
- Let uv be an arc, and split up the graph along this arc. Determine the spectrum of u and v recursively, and then join them together

Common spectrum, arc spectrum

- *Common spectrum*: $\tau_G : V^2 \to \mathbb{Z}_+^{n\times n}$
- $\tau_G(u, v)_{ij}$ is the number of topological orders in which the vertex u is in the i-th position, vertex v is in the j-th position.
- Special cases solved: the two vertices are connected with an arc (*arc spectrum*) or with a directed path.



Common spectrum, arc spectrum



Future plans

- Calculating the common spectrum of two vertices which are not connected by a directed path
- Calculating the common spectrum of $3, 4$ or arbitrary k

References

M. D. Atkinson.
On computing the number of linear extensions of a tree.
Order, 7(1):23–25, 1990.

G. Brightwell and P. Winkler.
Counting linear extensions is #P-complete.
In *Proceedings of the Twenty-Third Annual ACM Symposium on Theory of Computing*, pages 175–181, 1991.

K. Kangas, T. Hankala, T. M. Niinimäki, and M. Koivisto.
Counting linear extensions of sparse posets.
In *Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence*, pages 603–609, 2016.

K. Kangas, M. Koivisto, and S. Salonen.
A faster tree-decomposition based algorithm for counting linear extensions.
Algorithmica, 82(8):2156–2173, 2020.

Usage of AI tools

- Enchanting the code (mostly in logging)
- Writing small helper scripts in Python (e.g. drawing a tree based on it's edge list)