Directed studies - Eigenvalues of Cayley Graphs

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In this note, we present fundamental results concerning the eigenvalues of the transition matrix of probability measures defined on finite groups. Using these results, we then state relations concerning the eigenvalues of normal Cayley graphs. Then we show some applications of group growth and circulant graphs.

Fundamental theorems

First, let us see how one can define the Fourier transform of a function defined on a finite group using representations.

Definition 1 (Fourier transformation). Let ρ be any irreducible representation over $\mathbb{C}$ of the finite group G and $f : G \to \mathbb{C}$ a complex valued function on G (not necessarily a class function, which means that it is not necessarily constant on conjugacy classes). Then the Fourier transform of f at ρ is defined in the following way:

$$\widehat{f}(\rho) = \sum_{g \in G} f(g)\rho(g).$$

The next lemma is a consequence of Schur's lemma.

Lemma 1. (See [1, Lemma 5.]) Let G be a finite group, k the number of conjugacy classes in it $(C_1, C_2, \ldots, C_k)$, and $g_1, g_2, \ldots, g_k$ arbitrary class representatives. Let ρ be an irreducible representation of G and denote its character by χ_ρ. Take any complex valued class function f on G. Then $\widehat{f}(\rho)$ is a scalar matrix:

$$\widehat{f}(\rho) = \Big(\frac{1}{\chi_\rho(1)} \sum_{i=1}^{k} |C_i| f(g_i)\chi_\rho(g_i)\Big) I.$$

The following theorem characterizes the eigenvalues of an arbitrary transition matrix of a probability measure on a finite group.

Theorem 1. (See [1, Theorem 3.]) Let ρ be an irreducible representation of the finite group G, and P be a probability measure on G. Denote by $\mathrm{spec}(\rho)$ the spectrum of the matrix $\widehat{P}(\rho)$ and by $m(\lambda, \rho)$ the multiplicity of λ in $\mathrm{spec}(\rho)$. If $x_1, x_2, \ldots, x_{|G|}$ is an enumeration of the elements of G, then the transition matrix $M = \big(P(x_j x_i^{-1})\big)_{ij}$ has eigenvalues

$$\bigcup_{\rho \in \widehat{G}} \mathrm{spec}(\rho),$$

where an eigenvalue λ has multiplicity $\sum_{\rho \in \widehat{G}} \chi_\rho(1) m(\lambda, \rho)$.

Corollary 1. If we assume that P is a class function, then using Lemma 1, we find that for each ρ irreducible representation, there is an eigenvalue

$$\lambda_\rho = \frac{1}{\chi_\rho(1)} \sum_{i=1}^{k} |C_i| P(g_i)\chi_\rho(g_i)$$

with multiplicity $\chi_\rho(1)^2$.

The previous corollary is a general result that provides us with a formula for the eigenvalues of a normal cayley graph. Let S be a normal subset of the finite group G, which means that S is the union of some conjugacy classes of G. Let the probability measure P_S be the uniform distribution with support S, so

$$P_S(g) = \begin{cases} \frac{1}{|S|} & g \in S \\ 0 & g \notin S \end{cases}.$$

This way, the transition matrix corresponding to P_S is actually $\frac{1}{|S|}$ times the adjacency matrix of $Cay(G, S)$. Using Corollary 1, we conclude that the eigenvalues of the Cayley graph are

$$\lambda_\chi = \frac{1}{\chi(1)} \sum_{s \in S} \chi(s),$$

where $\chi \in \mathrm{Irr}(G)$. Moreover, the multiplicity of λ_χ is $\chi(1)^2$.

Applications

Let us see some applications of the above results. From now on, we assume that any generating set of a Cayley graph is normal and symmetric. Furthermore, to avoid self-loops, we also assume that $1 \notin S$. Using these results, one can prove a specific case of Babai's conjecture.

Conjecture 1 (Babai). There exist a universal constant C such that for all non commutative simple group G and for all its generating set S

$$\operatorname{diam}(Cay(G,S)) \leq (\log |G|)^C.$$

We make use of the character ratio $R(g) = \max\limits_{\substack{\chi \in \operatorname{Irr}(G) \\ \chi \neq 1_G}} \frac{|\chi(g)|}{\chi(1)}$ and state an upper bound for the second largest eigenvalue of a Cayley graph by absolute value.

Lemma 2. (See [5, Lemma 1.]) Let G be a finite group with a normal subset S, and let $\lambda\big(Cay(G,S)\big)$ be the second largest eigenvalue of $Cay(G,S)$ by absolute value. Then

$$\lambda\big(Cay(G,S)\big) \leq \max_{s \in S} R(s).$$

Moreover, if S is a conjugacy class, then we have an equality in the formula above.

With the help of this lemma and other results regarding the expansion of normal subsets, we can formulate the following:

Theorem 2. (See [5, Theorem 3.]) Let G be a finite simple group of Lie type, and let A be a nontrivial normal subset of G. Then either $|A^2| \geq |A|^{1+\varepsilon}$ or $G \setminus 1 \subseteq A^2$, for some constant $\varepsilon > 0$ depending on the rank of G only.

This is a stronger expansion result than the usual ones achieved in this area because this is a two step expansion result, unlike the usual three step ones (first introduced by Helfgott [2, Key Proposition.]). Finally, we take a look at the field of integral graphs.

Definition 2 (Integral graph). A graph is called integral if the spectrum of its adjacency matrix consists entirely of integers.

Let $\mathcal{D}_n$ be the divisors of n and $G_n(d) = \{k \mid 1 \leq k \leq n, (k,n) = d\}$, where d is a divisor of n.

Theorem 3. (See [6, Theorem 7.1.]) Let $S \subseteq \mathbb{Z}_n$. Then $Cay(\mathbb{Z}_n, S)$ is integral if and only if $S = \bigcup\limits_{d \in D} G_n(d)$ for some $D \subseteq \mathcal{D}_n$.

One direction is easy to see because we know what the characters of $\mathbb{Z}_n$ are, and $\mathbb{Z}_n$, being commutative, every subset of it is normal. So, using the derived formula for the eigenvalues of a normal Cayley graph, we get that

$$\lambda_k = \sum_{d \in D} \sum_{g \in G_n(d)} (\varepsilon^g)^k = \sum_{d \in D} c_{\frac{n}{d}}(k),$$

where $c_m(k) = \sum\limits_{g \in G_m(1)} (\varepsilon^g)^k$ is the Ramanujan sum with parameters m and k (ε is a primitive m^{th} root of unity). It is well known that $c_m(k)$ is always an integer, namely

$$\sum_{d \mid (m,k)} \mu\Big(\frac{m}{d}\Big) d.$$

Conjecture 2 (So). (See [6, Conjecture 7.3.]) Let $Cay(\mathbb{Z}_n, S_1)$ and $Cay(\mathbb{Z}_n, S_2)$ be two integral graphs. If $S_1 \neq S_2$, then $\operatorname{spec}(Cay(\mathbb{Z}_n, S_1)) \neq \operatorname{spec}(Cay(\mathbb{Z}_n, S_2))$.

In [3] Mikhail Klin and István Kovács proved that if $S_1 \neq S_2$, then $Cay(\mathbb{Z}_n, S_1) \ncong Cay(\mathbb{Z}_n, S_2)$ ([3, Corollary 11.2.]), which is weaker than So's conjecture because isomorphic graphs can have the same spectrum.

References

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