Random walks on graphs and electric networks

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1 Connections between electrical networks and random walks

At first glance it may not be clear how these two concepts relate to each other. But by observing the main properties of them the connection becomes apparent. A graph can be viewed as a network, whose edges are wires between the vertices and the transition propabilities between two vertices tell us how easy it is for an electron to travel through that edge compared to other edges with the same beginning. The corresponding physical property is the conductance of wires, which is the inverse of the resistance.
At this point we must point out that this analogy only works with reversible walks because the conductance of a cable is clearly the same regardless of which way a current goes through it.

Definition 1.1. A random walk on a graph $G = (V, E)$ with transition propabilities $p(x, y)$ is reversible if there exists a distribution π on the vertices such that $\pi(x)p(x, y) = \pi(y)p(y, x)$ for every $(x, y) \in E$.

Remark 1.2. We must observe that π does not have to be a probabilty measure on V. For example look at $\mathbb{Z}$ with the simple random walk. In this case $\pi \equiv 1$ works.

Remark 1.3. Note that π is unique up to multiplying with a positive constant as long as G is connected.

It is clear that the only way our analogy can work is if we define the conductance $c : E \longrightarrow \mathbb{R}_{\geq 0}$ of the edges from the reversible random walk as $c(x, y) = c(y, x) = \pi(x)p(x, y)$. Conversely, if we are given the conductances then the only way to define transition probabilities is by $p(x, y) = \dfrac{c(x, y)}{\sum_{x \sim y'} c(x, y')}$. The function $\pi : V \longrightarrow \mathbb{R}_{\geq 0}$ witnessing reversibility must be $\pi(x) = \sum_{x \sim y} c(x, y)$.
Now that we have established a connection let us use networks to help answer questions

about random walks.
An important question regarding infinite graphs is recurrence and transience. In the case of finite graphs what we are interested is that given two disjoint subsets A and B of V what is the probabilty that the random walk from $x \in V$ reaches A before B. In notation: $\mathbb{P}_x[\tau_A < \tau_B]$ where τ_A is the first time the random walk reaches A. The standard way of answering this is by considering the function $f(x) = \mathbb{P}_x[\tau_A < \tau_B]$ on V.
And observing that f is harmonic outside of $A \cup B$ i.e. $f(x) = \sum_{x \sim y} p(x,y)f(y)$. The following claim helps strengthen the connection with networks.

Claim 1.4. *If a function $g : A \cup B \longrightarrow \mathbb{R}$ is given then there exists a unique extension $\overline{g}$ to V which is harmonic at $V \setminus (A \cup B)$.*

Now we can use networks to solve this problem. Let $v|_A = f, v|_B = g$ a voltage function and extend it to V. Note that this voltage function is the same as $\mathbb{P}_x[\tau_A < \tau_B]$ if $f \equiv 1$ and $g \equiv 0$. Now we define the current i on the edges of G as $i(x,y) = c(x,y)[v(x) - v(y)]$. It can be shown that if $v|_A$ is any constant v_A and $v|_B \equiv 0$ then $\mathbb{P}_x[\tau_A < \tau_B] = \dfrac{v(x)}{v_A}$.
Let τ_A^+ be the first nonzero time the random walk reaches A and $A = \{a\}$. It is important wether we reach B or a first after leaving a. It can be calculated that

$$\mathbb{P}_x[\tau_B < \tau_a^+] = \mathbb{P}[a \to B] = \frac{1}{v(a)\pi(a)} \sum i(a,x).$$

So since $\sum i(a,x)$ is the current flowing into the network from a we can now calculate the effective resistance and conductance of the network. We know from physics that $C_{eff} = \dfrac{\sum i(a,x)}{v(a)}$ which means that $C_{eff} := \pi(a)\mathbb{P}[a \to B] =: \mathcal{C}(a \leftrightarrow B)$. So if we know the effective conductance we can calculate the probability of the random walk returning to a before hitting B.
Using this we can determine recurrence of infinite graphs if we know the effective conductance from a to infinity, where $C_{eff} = 0$ means recurrence. For example $\mathbb{Z}$ with the simple random walk corresponds to the network with constant 1 conductances. This means the resistances are also 1, and, as the edges are connected "in series", the resistances add up to an infinite effective resistance between 0 and infinity. This gives us that the effective conductance is zero, so the walk is recurrent.

2 Further applications of networks

By introducing the concept of energy we can show that transience is monotone i.e. if we increase the conductances a transient graph remains transient, furthermore a graph is transient if there exists a current with finite energy from a to infinity.
We can find many properties of weighted unifrom and fraction valued spanning trees using

conductances of edges and networks. We can also find maximal entropy weighted uniform spanning trees using Wilson's algorithm.

Next semester we are hoping to ecploit the electric network framework in cycle matroids of graphings and ask if fractional valued spanning forests of a graphing are convex combinations of $\{0, 1\}$ valued spanning forests, at least if the graphing is hyperfinite.