# Fundamentals of complex algebraic geometry

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# Preliminaries: Multi-dimensional complex analysis

Given a function $f : \mathbb{C}^n \to \mathbb{C}^m$ that is differentiable in the real sense, one can naturally extend complex analytical concepts to it.
One may treat $f$ as a real function:

$$f(x_1 + y_1 i, \ldots, x_n + y_n i) = f_{\mathbb{R}}(x_1, y_1, \ldots, x_n, y_n)$$

The real space $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ has an associated space of differential forms spanned by the natural basis $\{dx_i, dy_i\}$. In the complex perspective however, it is more natural to work with the basis $\{dz_i, d\bar{z}_i\}$ with:

$$dz_i = dx_i + i\, dy_i \qquad d\bar{z}_i = dx_i - i\, dy_i$$

For example, in the case with $1$ complex dimension, this notation is equivalent to the usual definition of the line integral:

$$\int_\gamma f(z)dz = \int_\gamma \mathsf{Re}(f(x + yi))dx + i \int_\gamma \mathsf{Im}(f(x + yi))dy$$

# Preliminaries: Wirtinger derivatives

$$dz_i = dx_i + i\, dy_i \qquad\qquad d\bar{z}_i = dx_i - i\, dy_i$$

This basis has a dual basis of directional derivatives. It is given by:

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\frac{\partial}{\partial x_i} - \frac{i}{2}\frac{\partial}{\partial y_i} \qquad\qquad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\frac{\partial}{\partial x_i} + \frac{i}{2}\frac{\partial}{\partial y_i}$$

These are called the **Wirtinger derivatives**. They are, again, quite natural. In the case with $1$ complex dimension, given a holomorphic function $f$, $\frac{\partial}{\partial z} f(z)$ is simply the complex derivative of the function, while $\frac{\partial}{\partial \bar{z}} f(z) = 0$. These may be derived from the Cauchy-Riemann equations.

# Preliminaries: Holomorphy

Working with the natural complex bases, the exterior derivative of a function becomes:

$$df = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial z_i} dz_i + \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i \right)$$

We say that a function is **holomorphic** if its exterior derivative is within the subspace spanned by $\{dz_i\}$. We say that it is **anti-holomorphic** if it is within the subspace spanned by $\{d\bar{z}_i\}$.

# Preliminaries: Holomorphic forms

We extend the definition of holomorphic functions to general differential forms:

$$d\omega = \sum_{I,J} d\omega_{I,J} \wedge dz_{I_1} \wedge d\bar{z}_{J_1} \wedge \ldots \wedge dz_{I_n} \wedge d\bar{z}_{J_n}$$

We say that the $p$-form $\omega$ is holomorphic if $d\omega$ is in $\wedge^{p+1}\text{span}\{dz_i\}$. This is equivalent to saying that $\omega$ is in $\wedge^{p}\text{span}\{dz_i\}$ and each of its coefficient functions are holomorphic.

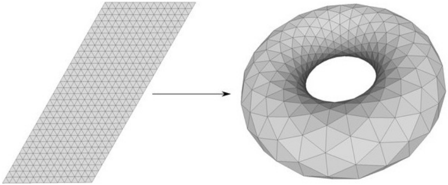
# Complex manifolds

## Definition (Complex manifold)

A **complex manifold** $M$ of dimension $n$ is a $2n$-manifold equipped with an open cover $\{U_i\}$ and charts $\{\varphi_i : U_i \to \mathbb{C}^n\}$ such that the transition maps $\varphi_i \circ \varphi_j^{-1}|_{U_i \cap U_j}$ are holomorphic.

Holomorphic functions are always smooth, so this is also a smooth manifold of dimension $2n$.

We may then talk about the usual real tangent spaces of a complex manifold at a point $p$, denoted as $T_{\mathbb{R},p}(M)$.



A torus may be supplied with a complex structure. All such structures can be realised as quotient spaces of the complex plane, up to biholomorphism.

# Complexified tangent space

## Definition

The **complexified tangent space** $T_{\mathbb{C},p}(M)$ at point $p$ is the vector space over $\mathbb{C}$ of directional derivatives.

A more concise way to write this is as the tensor product $T_{\mathbb{C},p}(M) = T_p(M) \otimes_{\mathbb{R}} \mathbb{C}$.
This just allows complex coefficients in directional derivatives.
So for example, if we have a scalar function $x^2 + y^2$ in local coordinates, then:

$$\begin{aligned}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(x^2+y^2) &= \left(\frac{\partial}{\partial x}x^2 + i\frac{\partial}{\partial y}x^2\right) + \left(\frac{\partial}{\partial x}y^2 + i\frac{\partial}{\partial y}y^2\right) \\ &= (2x + i\cdot 0) + (0 + i\cdot 2y) = 2x + 2iy\end{aligned}$$

# Holomorphic tangent space

Since holomorphic functions preserve holomorphy, we may also define the holomorphic tangent space:

## Definition (Holomorphic tangent space)

The **holomorphic tangent space** $T'_p(M)$ at a point $p$ is the subspace of the complexified vector space spanned by the directional derivatives $\{\frac{\partial}{\partial z_i}\}$.

This definition is independent of the chart chosen, since the pullback of the directional derivative $\frac{\partial}{\partial z_i}$ by a holomorphic transition map must itself lie in the subspace spanned by $\{\frac{\partial}{\partial z_i}\}$.

# Antiholomorphic tangent space

Similary,

## Definition

The **antiholomorphic tangent space** $T''_p(M)$ at a point $p$ is the subspace of the complexified vector space spanned by the directional derivatives $\{\frac{\partial}{\partial \bar{z}_i}\}$.

This is simply the complex conjugate of the holomorphic tangent space space. Clearly, the complexified tangent space decomposes as:

$$T_{\mathbb{C},p}(M) = T'_p(M) \oplus T''_p(M)$$

# The structure of differential forms

Denote with $A^p(M)$ the space of complex differential $p$-forms on $M$. By definition, this is the $p$-th exterior power of the cotangent space:

$$A^p(M) = \wedge^p T^*_{\mathbb{C}}(M)$$

Here, the cotangent space decomposes into the holomorphic and antiholomorphic cotangent spaces. Thus:

$$\begin{aligned} A^p(M) &= \wedge^p T^*_{\mathbb{C}}(M) \\ &= \wedge^p (T'^*(M) \oplus T''^*(M)) \\ &= \sum_{r+s=p} (\wedge^r T'^*(M) \oplus \wedge^s T''^*(M)) \end{aligned}$$

We denote $A^{r,s} := \wedge^r T'^*(M) \oplus \wedge^s T''^*(M)$, called the space of complex differential forms of form $(p,q)$.

# The exterior derivative

The exterior derivative $d : A^p \to A^{p+1}$ also extends to complex manifolds. It is not difficult to see that $\text{Im}(d|_{A^{p,q}}) \subset A^{p+1,q} \oplus A^{p,q+1}$. Thus the exterior derivative also decomposes into a holomorphic and antiholomorphic part:

$$d = \partial + \bar{\partial}$$

where

$$\partial|_{A^{p,q}} \subset A^{p+1,q} \quad \bar{\partial}|_{A^{p,q}} \subset A^{p,q+1}$$

It is then true that $d^2 = \partial^2 = \bar{\partial}^2 = 0$. These operators therefore both define cochains:

$$0 \longrightarrow A^0 \xrightarrow{\ d\ } A^1 \xrightarrow{\ d\ } A^2 \xrightarrow{\ d\ } \cdots$$

$$0 \longrightarrow A^{p,0} \xrightarrow{\ \bar{\partial}\ } A^{p,1} \xrightarrow{\ \bar{\partial}\ } A^{p,2} \xrightarrow{\ \bar{\partial}\ } \cdots$$

# Cohomologies

$$0 \longrightarrow A^0 \xrightarrow{\quad d \quad} A^1 \xrightarrow{\quad d \quad} A^2 \xrightarrow{\quad d \quad} \cdots$$
$$0 \longrightarrow A^{p,0} \xrightarrow{\quad \bar{\partial} \quad} A^{p,1} \xrightarrow{\quad \bar{\partial} \quad} A^{p,2} \xrightarrow{\quad \bar{\partial} \quad} \cdots$$

## Definition (De Rham cohomology)

The **de Rham cohomology** is given by $d$ acting on the spaces $A^p$:

$$H^p_{\mathsf{DR}} = \frac{\mathsf{ker}(d|_{A^p})}{\mathsf{im}(d|_{A^{p-1}})}$$

## Definition (Dolbeault cohomology)

The **Dolbeault cohomology** is given by $\partial$ acting on the spaces $A^{p,q}$:

$$H^{p,q}_{\bar{\partial}} = \frac{\mathsf{im}(\bar{\partial}|_{A^{p,q-1}})}{\mathsf{ker}(\bar{\partial}|_{A^{p,q}})}$$

# Example: De Rham cohomology of the torus

Given a (non-degenerate) lattice $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$, the quotient space $T = \mathbb{C}/\Lambda$ has the topology of a torus. It is also naturally a complex manifold. It has the de Rham cohomology groups of

$$H^0_{\mathsf{DR}}(T) = \mathbb{R} \qquad H^1_{\mathsf{DR}}(T) = \mathbb{R}^2 \qquad H^2_{\mathsf{DR}}(T) = \mathbb{R}$$

$H^0_{\mathsf{DR}}(T)$ represents that the space is connected.

$H^1_{\mathsf{DR}}(T)$ represents that a closed $1$-form is exact only if its integral is $0$ along any closed path. Since the fundamental group of $T$ has $2$ dimensions, so does $H^1_{\mathsf{DR}}(T)$. It is generated by $dx$ and $dy$.

$H^2_{\mathsf{DR}}(T)$ represents that a closed $2$-form is exact only if its integral is $0$ along the entire space. As such, $H^2_{\mathsf{DR}}(T)$ is generated by $dx \wedge dy$.

All other cohomology groups are trivial, since there are no non-zero 3-forms on $T$.

# Example: Dolbeault cohomology of the torus

The Dolbeault cohomology groups of the torus are

$$H^{0,0}_{\bar{\partial}}(T) \simeq \mathbb{C} \qquad H^{0,1}_{\bar{\partial}}(T) \simeq \mathbb{C}$$
$$H^{1,0}_{\bar{\partial}}(T) \simeq \mathbb{C} \qquad H^{1,1}_{\bar{\partial}}(T) \simeq \mathbb{C}$$

These happen to correspond nicely to the de Rham cohomology groups. It turns out that the groups have the following generators:

$$H^{0,0}_{\bar{\partial}}(T) = \langle 1 \rangle \qquad H^{0,1}_{\bar{\partial}}(T) = \langle d\bar{z} \rangle$$
$$H^{1,0}_{\bar{\partial}}(T) = \langle dz \rangle \qquad H^{1,1}_{\bar{\partial}}(T) = \langle dz \wedge d\bar{z} \rangle$$

# Example: Dolbeault cohomology of the torus

$$H^{0,0}_{\bar\partial}(T) = \langle 1 \rangle \qquad H^{0,1}_{\bar\partial}(T) = \langle d\bar z \rangle$$
$$H^{1,0}_{\bar\partial}(T) = \langle dz \rangle \qquad H^{1,1}_{\bar\partial}(T) = \langle dz \wedge d\bar z \rangle$$

So actually,

$$H^1_{\mathsf{DR}}(T) \otimes \mathbb{C} = H^{1,0}_{\bar\partial}(T) \oplus H^{0,1}_{\bar\partial}(T)$$

In other terms, the de Rham cohomology decomposes into the appropriate levels of the Dolbeault cohomology. This is called the Hodge decomposition. This is the case for all compact so-called Kähler manifolds. In this context, the Dolbeault cohomology refines the de Rham cohomology.

# Example: The punctured plane

The punctured plane, $\mathbb{C}^* = \mathbb{C} \backslash \{0\}$, is also a complex manifold. $H^1_{\mathsf{DR}}(\mathbb{C}^*)$ is 1-dimensional, generated by the form

$$\frac{x\,dy - y\,dx}{x^2 + y^2}$$

This is the derivative of the argument function.

$H^0_{\mathsf{DR}}(\mathbb{C}^*)$ is also 1-dimensional, and higher cohomology groups are trivial.

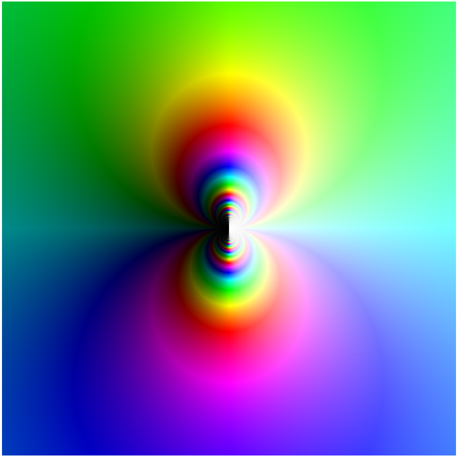
# Example: The punctured plane

In contrast, the Dolbeault cohomology is not so nice. Since $\bar{\partial}$ takes all holomorphic functions to $0$, $H^{0,0}_{\bar{\partial}}(\mathbb{C}^*)$ consists of all holomorphic functions. As a vector space, this has countably infinite dimensions. So we see that Hodge decomposition fails for non-compact spaces.

$H^{1,0}_{\bar{\partial}}(\mathbb{C}^*)$ is also infinite dimensional, consisting of all holomorphic forms.

$$H^{0,0}_{\bar{\partial}}(\mathbb{C}^*) = \{f : \mathbb{C}^* \to \mathbb{C} \text{ holomorphic}\}$$
$$H^{1,0}_{\bar{\partial}}(\mathbb{C}^*) = \{f\, dz \mid f : \mathbb{C}^* \to \mathbb{C} \text{ holomorphic}\}$$

# Example: The punctured plane



The scalar function $\exp(-1/x)$ is holomorphic on the punctured plane, but it is not $0$.

# Example: The punctured plane

All other Dolbeault cohomology groups are trivial. For example, for any holomorphic $f$, the $1$-form $f\,dz$ is exact, since

$$\begin{aligned}\bar{\partial}(f\bar{z}) &= \bar{z}\bar{\partial}(f) + f\bar{\partial}(\bar{z}) \\ &= 0 + f\end{aligned}$$

Even the classical obstruction of $\frac{1}{\bar{z}}$ not being integrable disappears.

$$\begin{aligned}\bar{\partial}\log|z|^2 &= \bar{\partial}(\log(z) + \log(\bar{z})) \\ &= 0 + \frac{1}{\bar{z}}\end{aligned}$$

So all smooth functions on $\mathbb{C}^*$ are $\bar{\partial}$-exact, which is why the higher Dolbeault cohomology groups are trivial.

# Poincaré lemma

To help us in the study of these cohomologies, we have the Poincaré lemma.

## Lemma (Poincaré)

*Let $\omega \in A^p(\Delta)$ for a contractible set $\Delta$ for which $d\omega = 0$. Then $\exists \varphi \in A^{p-1}(M) : d\varphi = \omega$.*
*Ergo, all closed $p$-forms are exact.*

## Lemma ($\bar{\partial}$-Poincaré)

*Let $\omega \in A^{p,q}(\Delta)$ for a contractible set $\Delta$ for which $\bar{\partial}\omega = 0$. Then $\exists \varphi \in A^{p,q-1} : \bar{\partial}\varphi = \omega$.*
*Ergo, all closed $(p, q)$-forms are exact.*

These tell us that the de Rham and Dolbeault cohomologies are locally trivial.

# Aside: The de Rham theorem

Suppose that $\sigma$ is a closed singular $p$-chain and $\omega$ a closed $p$-form. $p$-chains can be smoothly approximated. Thus we can calculate the integral:

$$I_\omega(\sigma) = \int_\sigma \omega$$

This is linear in $\omega$ and $\sigma$. If $\sigma$ is a boundary, or if $\omega$ is exact, then by the Gauss theorem, $I_\omega(\sigma) = 0$. The construction thus induces a morphism:

$$H_{\mathsf{DR}}(M) \to H^*_{\mathsf{sing}}(M)$$
$$\omega \mapsto I_\omega$$

## Theorem (de Rham)

*This is an isomorphism.*

# The Dolbeault theorem

One might think to ask if the de Rham theorem has an analogue for the Dolbeault cohomology. As it turns out, it does, via the **Čech cohomology**. Denoting the sheaf of holomorphic $p$-forms $\Omega^p$, the following holds:

## Theorem (Dolbeault)

$$H^{p,q}_{\bar{\partial}} \simeq \check{H}^q(M, \Omega^p)$$

This may be proven using a sufficiently fine covering to make every sheaf section exact and then using $\bar{\partial}$-Poincaré lemma and the zig-zag lemmma to repeatedly lower the order of the Čech cohomology group while raising the number of antiholomorphic dimensions in the sheaf. The right hand side eventually reduces to the definition of $H^{p,q}_{\bar{\partial}}$.

# AI disclosure

ChatGPT (Free plan) was used during research.

No other AI tools were used.