Sandpile group actions

Introduction

In this paper, we will examine the sandpile group. First, we introduce the main definitions and theorems regarding the group. Then we take a closer look at the rotor-routing action and consistency. Consistency means that in some cases the action stays the same when we delete or contract an edge. It is known that for planar graphs the rotor-routing action is consistent. There are multiple results showing consistent sandpile group actions, but so far there was no result showing that consistency fails for some naturally defined action. In this project we show an example for this. We introduce consistency on non-planar graphs and examine it.

Sandpile group

Let G be a connected, undirected graph and $c \in \mathbb{Z}^V$ chips on the vertices. Such a c is called a *chip configuration*. There can be negative chips on a vertex as well. We denote the number of chips on a vertex v in c by $c(v)$. These configurations form an additive group. Take the subgroup for which the sum of the elements of c is 0. This also forms an additive group denoted by $\mathbb{Z}_0^V$, with the identity element $\underline{0} \in \mathbb{Z}^V$.

We introduce an equivalence between the configurations of the chip-firing game. In the *chip-firing game*, in one step, we choose a vertex v and fire a chip to each of its neighbors. After the step, the number of the chips on the chosen vertex decreased by the degree, and on every neighbouring vertex it increased by one. We can fire a negative (integer) number of chips from v too. Notice, that for simple graphs it is the same as if we did a chip firing step from all of the vertices except v. Here, all vertices non-adjacent to v fire one chip and receive one chip through all of the incident edges. Every u neighbor of v fires d_u chips (where d_u is the degree of u) and receives back a chip from every adjacent vertex except v. So the number of chips on v increases by d_v. Two configurations are *equivalent* if they can be reached from each other with chip-firing steps. Note that it is the same definition if we say that c and c' are equivalent if there exists some $x \in \mathbb{Z}^V$ such that $c + L_G x = c'$, where L_G is the Laplacian matrix of G. Denote the equivalence with $\sim_G$. The $\mathbb{Z}_0^V / \sim_G$ quotient set is the sandpile group denoted by $S(G)$. The number of the elements of $S(G)$ is equal to the number of the spanning trees of G.

There is a group action of the sandpile group on the spanning trees of G. To describe this action first, we introduce *rotor-routing*. We fix a cyclic order of the incident edges for all of the vertices in G. This is called a ribbon structure. For a vertex v and an incident edge e, we denote by $f_v(e)$ the edge following e at vertex v.

Let us call a planar graph embedded in the plane a plane graph. Given a plane graph, the counter-clockwise ordering around the vertices generates a ribbon structure. Let us denote this plane graphs with the ribbon structure a plane ribbon graph. We will mostly work with plane ribbon graphs.

It is known that for any G' graph and ribbon structure exists an orientable surface in which it can be embedded.

We describe the rotor-routing for plane ribbon graphs. Suppose that we have a spanning tree T. Firstly, we define the rotor-routing algorithm with root t for a chip configuration such that there is 1 chip on $s \in V(G)$, -1 chip on $t \in V(G)$, and 0 chip on all other vertices. First we orient every edge of T towards t. There is a unique path from every vertex of G to t in the tree, so we can orient all these paths' edges towards t. This way we obtained a spanning in-arborescence $\overline{T}$ with root t. In $\overline{T}$ every vertex has out-degree 1, except for the root t. This way we assign an edge to all vertices except t. Denote this edge (so called *rotor*) assigned to

v by $p(v)$. We call this p function a rotor configuration. From now on, in every step i we take the vertex with the chip v_i and $p(v_i)$. We take the next edge $p(v_i) := f_{v_i}(p(v_i))$ from v_i (we rotate $p(v_i)$), and move the chip from v_i to the head of $p(v_i)$. If the chip arrives at t, then we stop, otherwise we repeat the step (rotate the edge and move the chip). It can be shown that the algorithm ends in finitely many steps, and at the end we obtain a $\overline{T}_2$ in-arborescence with root t.

So far, we have defined the action for a $c^* \in S(G)$ where c^* has one 1 and one -1 element (at the root) and all of the others are 0. We can define the action for chip configurations, where only the root has negative chips. $S(G)$ is an additive group, so every $c \in S(G)$ can be written as the sum of $c_k^* \in S(G)$, where c_k^* has $|V| - 2$ zero elements, one 1, and one $c_k^*(t) = -1$. Thus we can determine the action of any $c \in S(G)$ as the composition of the actions of c_k^*. It has been proven, that for equivalent configurations the action is the same.

We noted earlier that we show the rotor-routing with root t, towards which we oriented all the edges of the tree. In the general case, root can be any other $r \in V(G)$.

Next we show the rotor-routing algorithm for $r \neq t$. In this case, the first step is to find a c' equivalent to c_{st} such that in c' every vertex has non-negative chips except for r. There is always a chip configuration like this (it can be obtained by firing a lot of chips from the root and the vertices close to it). Then we can write c' as the sum of c_j^* configurations such that for all j the -1 chip belongs to r. Now we can perform the standard rotor-routing algorithm for these c_j^* and root r. The composition of these actions is the action of c_{st}.

It has been shown, that if G is a plane ribbon graph, then the rotor-routing action is independent of the choice of the root. For non-plane graphs, the action depends on the root.

Let G be an undirected graph on 8 vertices embedded in the plane as on the figure and the ribbon structure be the counter-clockwise order of the edges around the vertices except for v_4. At v_4 the cyclic order of edges would be $v_4v_2 - v_4v_3 - v_4v_6$ on the plane, but we will use $v_4v_2 - v_4v_6 - v_4v_3$. This ribbon structure cannot be embedded in the plane since it has a cycle $v_1v_2v_4v_6v_5v_3$ and an edge, that goes from v_3 into the cycle and arrives at v_4 from the outside.

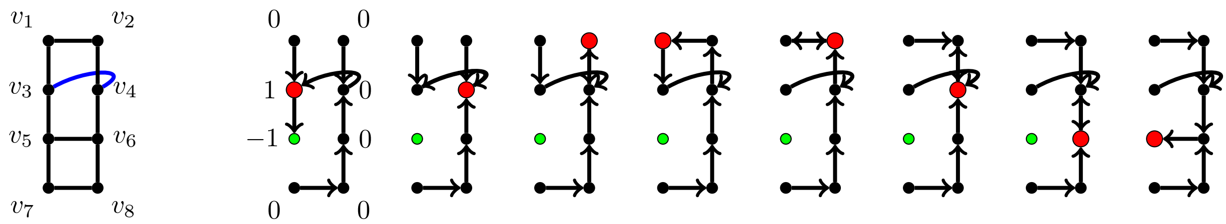


Figure 1: Non-planar graph and rotor-routing

In Figure 1 we show how the rotor-routing algorithm with t root works for $s = v_3$ and $t = v_5$ and the tree T'. The edges of T' are v_1v_3, v_3v_5, v_3v_4, v_2v_4, v_4v_6, v_6v_8 and v_7v_8.

Consistency

One important property of the sandpile torsor algorithm on spanning trees of plane ribbon graphs is the consistency. This means that the algorithm works similarly when we delete or contract an edge. We denote by $G \backslash e$ when we delete e from G and by G/e when we contract edge e in graph G. If G is a plane graph, then G/e and $G \backslash e$ are plane graphs too. Let $\alpha_G(s - t, T)$ denote the tree that we get by performing the rotor-routing algorithm on T tree with $c(s) = 1$, $c(t) = -1$ where t is the root and $st \in E(G)$.

Definition 0.1. *A sandpile torsor algorithm is consistent if for every plane ribbon graph, every T tree and every $s, t \in V(G)$ (such that $st \in E(G)$) the following properties hold*

- *For any $e \in E(G)$ where $e \neq st$ if $e \in T$ and $e \in \alpha_G(s-t, T)$:*
$$\alpha_{G/e}(s-t, T \backslash e) = \alpha_G(s-t, T) \backslash e$$
- *For any $e \in E(G)$ if $e \notin T$ and $e \notin \alpha_G(s-t, T)$:*
$$\alpha_{G \backslash e}(s-t, T) = \alpha_G(s-t, T)$$
- *For any $e \in E(G)$ such that $e \neq st$, if there exists x cut vertex, then*
$$e \in T \Longleftrightarrow e \in \alpha_G(s-t, T)$$

The rotor-routing is consistent for every plane ribbon graph[1]. The proof relies on the fact that the rotors do not make more than one turn during the algorithm. Furthermore, if we take the directed cycle consisting the unique path between s and t and the ts edge, then the chip always stays on one side of this cycle (with clockwise orientation it is the right side). This is the reason that the proof only applies to plane ribbon graphs since in other graphs a cycle not necessarily divides the surface into two parts.

In Figure 2 and 3 we can see an example for the second condition of the consistency on plane graphs. Let G' be the same as the G graph shown in Figure 1, except that at v_4 take the counter-clockwise cyclic ordering of the edges. This way we obtain a plane ribbon graph. The $\alpha_{G'}(v_5 - v_6, T)$ rotor-routing algorithm generates the following cycle of the trees. This is the orbit of the initial tree with respect to the subgroup of $S(G)$ generated by $c(v_5) = 1$, $c(v_6) = -1$ and $c(v_i) = 0$ for $i = 1, 2, 3, 4, 7, 8$.

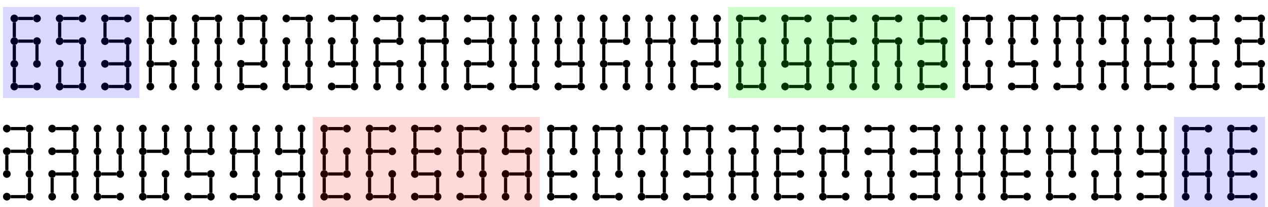


Figure 2: The trees generated by $\alpha_{G'}(v_5 - v_6, .)$

Now we delete an edge $e = v_2v_4$ and show in Figure 3 the trees generated by $\alpha_{G' \backslash e}(v_5 - v_6, T')$. Here we obtain 3 orbits with five trees each.

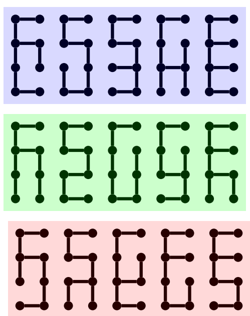


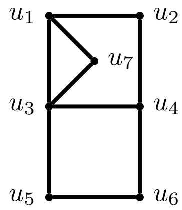
Figure 3: The trees generated by $\alpha_{G' \backslash e}(v_5 - v_6, .)$.

As we can see on the colored trees, if $e \notin T'$ and $e \notin \alpha_{G'}(v_5 - v_6, T')$, then $\alpha_{G'}(v_5 - v_6, T') = \alpha_{G' \backslash e}(v_5 - v_6, T')$.

[1] Ankan Ganguly and Alex McDonough, *Rotor-routing induces the only consistent sandpile torsor structure on plane graphs*, arXiv:2203.15079, 2022.

This way the consistency reveals that the action *behaves well* on plane ribbon graphs. The question may arise whether the rotor-routing algorithm behaves well on some non-plane graphs too. For this we need to define consistency for non-plane graphs. Here we take the surface in which the graph with the ribbon structure can be embedded. When we contract an edge, the cyclic ordering around the new vertex (created by merging the two endpoints) is the clockwise orientation of the incident edges on the surface. It is the same as if we concatenated the two edge lists to get the cyclic ordering. When we delete an edge then we simply restrict the cyclic ordering to the remaining edges. This way the definition is almost the same as in planar case, the difference is that we set a root in the definition to determine the action. We will examine actions on non-plane graphs. Below we show a counterexample to consistency.

Let G^* be a graph on 7 vertices as shown on the right. $E(G^*) = \{u_1u_2, u_1u_3, u_1u_7, u_2u_4, u_3u_4, u_3u_7, u_3u_5, u_4u_6, u_5u_6\}$. At u_2, u_3, u_5, u_6, u_7 the edges are in counter-clockwise order. Around u_1 and u_4 the edges are in clockwise order.



In Figure 4 there are the trees generated by the rotor-routing algorithm with root u_7 on G^*, where $c(u_1) = 1$, $c(u_7) = -1$. Since for non-plane graphs the action depends on the root, we modify the notation to mark the root like $\alpha^{u_7}_{G^*}(u_1 - u_7, T_1)$.

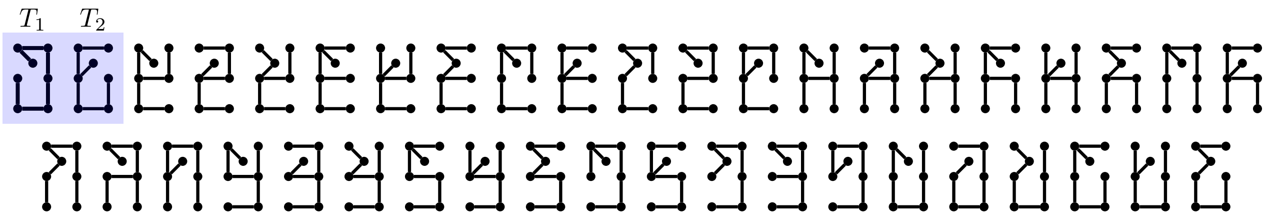


Figure 4: The orbit generated by $\alpha^{u_7}_{G^*\setminus e}(u_1 - u_7, .)$

In Figure 5, there are the trees obtained when we run the algorithm after deleting the u_3u_4 edge from G^*. As we can see from the colored trees, consistency does not apply. It can be easily verified that $u_3u_4 \notin T_1$ and $u_3u_4 \notin \alpha^{u_7}_{G^*}(u_1 - u_7, T_1)$, but $\alpha^{u_7}_{G^*}(u_1 - u_7, T_1) = T_2 \neq \alpha^{u_7}_{G^*\setminus e}(u_1 - u_7, T_1)$.

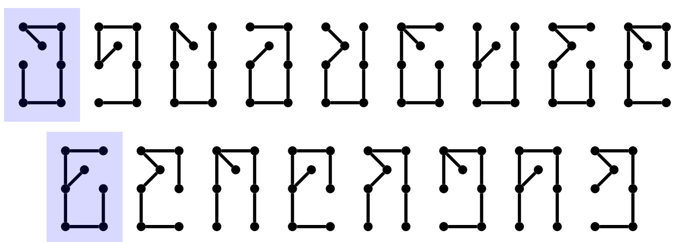


Figure 5: The orbit generated by $\alpha^{u_7}_{G^*\setminus e}(u_1 - u_7, .)$ with deleted u_3u_4

We can see that non-plane graphs are not always consistent. It would be interesting to examine which graphs and actions are consistent and if it depends on the root of the action. Note that for planar graph the rotor-routing was independent from the root, thus we assumed it to be the vertex with -1 chip. For non-plane graph the action depends on the root.

We were examining the case when we delete an edge, such that without that edge the graph becomes a plane ribbon graph, too. For this we used the previously described G graph and its ribbon structure and deleted one of the incident edges of v_4. With this modification the degree of v_4 is only 2, so every cyclic ordering of the incident edges is the same. It turned out that at least in the observed cases the rotor algorithm was consistent. Moreover, we looked at the action with other deleted edges and some roots other than the vertex with -1 chip, but these were also consistent.

Another question would be to find out if the action is consistent for a graph and ribbon structure where after deleting or contracting the edge the surface does not change. Note that the example in Figure 4 and 5 answers this question. Both the original graph with the ribbon structure and the new graph obtained with deleting an edge can be embedded in the same surface: a torus, and none of them can be embedded in the plane.