Positional games

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I am reading the book Positional games from authors Dan Hefetz, Michael Krivelevich, Miloš Stojaković and Tibor Szabó. One of the authors, Miloš Stojaković gave a minicourse on positional games here at ELTE and I became interested in positional games, that's why I chose it as the subject of my Directed studies. I read the first three chapters from the book so far and worked on the examples presented at the end of each chapter. These first three chapters in order are Introduction, Maker-Breaker Games and Biased Games.

1 INTRODUCTION

The authors note that combinatorics (in particular Combinatorial Games) is a very concrete science, that's why they begin by discussing, concrete, easy-to-understand examples of combinatorial games. The first example is Tic-Tac-Toe and its 2-dimensional generalizations and the authors introduce the notion of *pairing draw*, which means that we assign the possible moves into pairs and claiming the sibling of the just claimed element as the second player after any move of the first player leads to a draw. The classical 3×3 Tic-Tac-Toe is a draw by case analysis, the 4×4 case becomes a pairing draw after a suitable first move of the second player and on every board of size at least 5×5, the game is a pairing draw.
The second example is n^d, which are the d-dimensional generalizations of Tic-Tac-Toe, which we know little about. For fixed n, the first player wins for large enough d and for fixed d, the game becomes a draw and even a pairing draw for large enough n, but we do not know much m,ore than this. The winning strategy as the first player for the game 4^3 (marketed as Qubic) has the size of a phone book as one of the solvers, Oren Patashnik put it, which describes the complexity of these generalizations. The other examples are Hex, for which John Nash came up with the idea of strategy stealing, Connectivity game, Sim, Hamiltonicity game and Row-coloumn game.

After looking at these examples, the authors define positional games in a general setting. A positional game is played on a set X (usually finite), the board of the game and the two players take turns occupying previously unoccupied elements of the board. In the most general version, there are two parameters p, q, the first player takes p unoccupied elements of X and the second player takes q unoccupied elements in his turn. There is a given set of subsets, which the players focus on. A "Maker" player tries to claim a whole winning set, "Breaker" player tries to prevent the opposing "Maker" player from claiming a winning set. An "Avoider" player tries to not claim a whole losing set and an "Enforcer" tries to prevent the opposing "Avoider" player. The four main types of studied positional games are Maker-Maker (The first player who claims a whole winning set wins), Maker-Breaker, Avoider-Enforcer and Avoider-Avoider. The game ends in a draw if none of the Maekrs succeed or both of the Avoiders succeed.

There are three possible results of these games, when both players are computationally powerful, one of the players has a winning strategy or both of the players have a non-losing strategy. When played on a finite board, every positional game can be described by a tree of all possible moves, the so-called game-tree and we can always solve the game by backtracking from the leaves, every player always choosing the best possible alternative for him. It sounds simple, however the game tree is huge even for small games, for example, recall the phone book-sized winning strategy for Qubic or 4^3. This leaves plenty of room to develop tools and theory for positional games, which are computable, we usually want polynomial algorithms instead of computing the exponentially sized game tree. Positional games are two-player perfect information games in the terms of classical Game Theory, where the focus is usually on imperfect information giving rise to probabilistic arguments and mised strategies. They can be solved by exhaustive search, thus they are considered to be trivial in classical Game Theory, but in reality this turns out to be computationally false.

The author's conclude the chapter by stating the most important known results for Maker-Maker games, the so-called *strong* games, where both players try to occupy a whole winning set before the other player, for example Tic-Tac-Toe and its generalizations, n^d are strong games. In principle, there are three possible results of these games, but Second Player's win is impossible due to *strategy stealing*. Let's assume Second Player has a winning strategy, which is a playbook for how to play for every given move of the First Player. If first Player follows this Playbook, after beginning with a random move. The only problem could arise when the strategy would require to play the first random move, but then the First Player makes another arbitrary move. Thus, this strategy would be a winning strategy for the First Player, leading to a contradiction, making Second Player's win theoretically impossible. This is a pretty amazing result due to the applicability for all strong games, however in real examples it cannot be used as it is an existence proof. Strategy stealing is most powerful when combined with a Ramsey-type result. A Ramsey-type result for a given game says that in every possible final position, at least one of the players occupies a winning set fully, making a draw impossible, therefore that game is First Player's win. The authors mention that the most famous example for this is the n^d game, where Hales and Jewett proved the Ramsey-type result: for every large enough $d \geq d_0(k, n)$, every k-coloring of n^d contains a monochromatic line. There are many open problems regarding strong games, for example giving explicit drawing stretegies in general for the First Player and giving explicit winning strategies for First Player in Ramsey-type games.

2 MAKER-BREAKER GAMES

In Maker-Breaker type games, the Breaher is no longer concerned with trying to make a winning set of his own, his only goal is to occupy one element of each winning set, therefore making Maker's win impossible. Usually we set Maker as First Player and Breaker as Second Player. Being first is an advantage, the following statements hold for every game. If Maker wins as Second Player, he also wins as First. Similarly if Breaker wins as Second Player, he also wins as First. Studying Maker-Breaker games and strong games complement each other. If Breaker (as Second Player) can win in a Maker-Breaker game, then the corresponding strong game is a draw, by Second Player following the Breaker strategy. Similarly, if First Player can win in the strong version, then Maker (as First) can win in the corresponding Maker-Breaker game.
This chapter focuses on the Erdős-Selfridge criterion and its applications to various games. The Erdős-Selfridge criterion states that for a set of winning sets $\mathscr{F}$, then if $\sum_{A\in\mathscr{F}} 2^{-|A|} < \frac{1}{2}$, then the game is Breaker's win. The proof defines the danger of a hypergraph $\mathscr{F}'$ as $\sum_{A\in\mathscr{F}'} 2^{-|A|}$, where $\mathscr{F}'$ is the hypergraph we get after removing every set from $\mathscr{F}$ from which Breaker has already claimed an element and removing every element claimed by maker from the sets. Breaker's strategy is to decrease the danger as much as he can. If he plays this way, the danger cannot increase after a pair of Breaker-Maker moves. Maker can at most double the danger with his first move, therefore it remains below 1 the whole game. Maker's win would mean a danger value of at least 1 in the final position, thus the described strategy is a winning strategy for Breaker. The authors apply this criterion for 2-colorings of graphs, the clique game and the n-in-a-row game.

3 BIASED GAMES

This chapter focuses on Maker-Breaker games where if the player's take turns alternately claiming only one element, one of them has an easy winning strategy. This leads us to the investigation of biased games, where there are two parameters p, q, the Maker takes p unoccupied elements of X and Breaker takes q unoccupied elements of X in his turn.
The main result of this chapter is the generalized version of the Erdős-Selfridge theorem, which states the following for the $p : q$ biased Maker-Breaker games. If $\sum_{A\in\mathscr{F}} 2^{-|A|/p} < \frac{1}{1+q}$, then Breaker (as Second Player) has a winning stretegy for the game. The authors apply this result to the Connectivity game, where Maker's goal is to claim a connected graph on n verticesand they give a probabilistic heuristic for determining a winning threshold for Maker-Breaker games on graphs, using random graphs.