Stability properties of Runge–Kutta methods

Temesvári Ádám
Supervisor: Havasi Ágnes

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Introduction: A-stability

Definition (Dahlquist's test equation)

$$\begin{cases} y'(t) = \lambda y(t) & \qquad \lambda \in \mathbb{C} \\ y(0) = 1 \end{cases}$$

Applying a RK method for the test equation we get

$$y_{n+1} = R(z) y_n$$
$$z = \lambda h \in \mathbb{C}$$

Definition (A-stability)

An s-stage RK method is called A-stable if its stability function satisfies $|R(z)| \leq 1$ *for all* $z \in \mathbb{C}^-$.

AN-stability

$$\begin{cases} y'(t) = \lambda(t)y(t) & \lambda\colon \mathbb{C} \to \mathbb{C} \\ y(0) = 1 \end{cases}$$

Let $Z = \text{diag}(z_1, \ldots, z_s) \in \mathbb{C}^{s \times s}$, where $z_i = h\lambda(t_n + hc_i)$. Then y_{n+1} can be expressed as:

$$y_{n+1} = (1 + b^\intercal Z(I - AZ)^{-1}\mathbb{1})y_n$$

Definition (AN-stability)

An s-stage RK method is called AN-stable if the function

$$K(Z) = 1 + b^\intercal Z(I - AZ)^{-1}\mathbb{1}$$

satisfies $|K(Z)| \leq 1$, *with* $Z = \text{diag}(z_1, \ldots, z_s)$, *where* $z_i \in \mathbb{C}^-$ *for all* $i = 1, 2, \ldots, s$, *and* $z_i = z_j$, *when* $c_i = c_j$.

AN-stability, Remarks

Remark

If $\lambda(t) = \lambda$*, then* $Z = h\lambda I$*, and we recover the stability function for the autonomous scalar case.*

This implies the following statement:

Corollary

If an s-stage RK method is AN-stable, then it is A-stable.

Remark

An $s = 1$ *stage RK method is AN-stable if and only if it is A-stable.*

B/BN stability

For B/BN stability we leave behind Dahlquist's test equation and consider autonomous/non-autonomous systems.

Definition (Dissipative system)

The system $y'(t) = f(t, y(t))$ *is dissipative on* $[a, b]$ *if*

$$\langle f(t, y(t)) - f(t, \tilde{y}(t)), y(t) - \tilde{y}(t)\rangle \leq 0$$

holds $\forall t \in [a, b]$ *where* $y(t)$ *and* $\tilde{y}(t)$ *are solutions starting from different initial conditions.*

B/BN stability

Definition (Contractive solution)

Let $y(t)$ *and* $\tilde{y}(t)$ *be two solutions for* $y'(t) = f(t, y(t))$ *with initial values* $y(a) = \eta$, $\tilde{y}(a) = \tilde{\eta}$, $\eta \neq \tilde{\eta}$. *If*

$$||y(t_2) - \tilde{y}(t_2)|| \leq ||y(t_1) - \tilde{y}(t_1)||$$

holds for all $t_1 \leq t_2$ *such that* $a \leq t_1 \leq t_2 \leq b$, *then the solutions of the system are called contractive on* $[a, b]$.

B/BN stability

Proposition

If the system $y'(t) = f(t, y(t))$ is dissipative, then its solutions are contractive.

Finally, we need to define the contractivity of the numerical method/solution.

Definition (Contractive numerical solution)

Let $\{y_n\}$ and $\{\tilde{y}_n\}$ be any two numerical solutions obtained by a RK method for different initial conditions. If

$$||y_{n+1} - \tilde{y}_{n+1}|| \leq ||y_n - \tilde{y}_n||$$

holds for $0 \leq n \leq N$, then the numerical solutions and the numerical method are contractive on $t \in [t_0, t_N]$.

B/BN stability

Definition (B-stability)

A RK method is B-stable if for any stepsize $h > 0$ the method gives contractive solutions when applied to a dissipative autonomous system.

Note that B-stability only refers to autonomous systems, so it yields a nonlinear generalization of A-stability.

Proposition

If a RK method is B-stable, then it is A-stable.

B/BN stability

Theorem (Butcher and Burrage)

Let B and Q be $s \times s$ matrices defined as

$$B := \mathit{diag}(b_1, \ldots, b_s)$$

$$Q := BA^{-1} + (A^{-1})^\intercal B + (A^{-1})^\intercal b b^\intercal A^{-1}.$$

Then if B and Q are positive semidefinite matrices, the method is B-stable.

Definition (BN-stability)

A RK method is BN-stable if for any stepsize $h > 0$ the method is contractive when applied to a dissipative system.

Proposition

If a RK method is BN-stable, then it is B-stable and AN-stable.

Algebraic stability

Butcher and Burrage showed a sufficient condition for B and BN-stability, called *algebraic stability*.

Definition (Algebraic stability)

Let B and M be $s \times s$ matrices defined as

$$B := \mathit{diag}(b_1, \ldots, b_s), \quad M := BA + A^\intercal B - bb^\intercal.$$

Then if B and M are positive semidefinite matrices, the method is called algebraically stable.

Theorem

If a method is algebraically stable, then it is B and BN-stable.

Nonconfluent RK methods

Definition

A RK method is nonconfluent if $c_i \neq c_j$ *when* $i \neq j$ *and confluent if* $\exists\, i \neq j$ *such that* $c_i = c_j$.

For nonconfluent methods we can state the following two statements regarding stability relations:

Theorem

If a RK method is algebraically stable, then it is AN-stable and, if the method is nonconfluent, then the converse also holds, i.e., an AN-stable method is algebraically stable.

Theorem

A nonconfluent RK method is AN-stable if and only if it is BN-stable.

Summary

Theorem

For a nonconfluent RK method the following implications hold:

$$\left.\begin{cases} \text{Algebraic stability} \\ \Updownarrow \\ \text{BN-stability} \\ \Updownarrow \\ \text{AN-stability} \end{cases}\right\} \Longrightarrow \text{B-stability} \Longrightarrow \text{A-stability}.$$

Summary

Theorem

For a genereal RK method the following implications hold:

$$\textit{Algebraic stability} \Longrightarrow \left\{ \begin{array}{ccc} \textit{BN-stability} & \Rightarrow & \textit{B-stability} \\ & & \Downarrow \\ \textit{AN-stability} & \Rightarrow & \textit{A-stability} \end{array} \right\}.$$

Thank you for your attention!

AI usage

I, Temesvári Ádám, declare that I used ChatGPT-4o for styling and writing parts of the LaTeX code for this presentation.